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Biographies of the contributors in this issue will appear in the next issue.

COMPACTNESS IN THE WEAK TOPOLOGY

John Wells Brace

1. *Foreword.* In topological spaces there is a very important class of sets called the compact sets. On the real line with the usual topology a closed interval of finite length is a specific example of such a set. In the course of development several definitions for compact sets appeared. In metric spaces such as the real line the definitions are equivalent, while in more general topological spaces the relationship is not always known. This paper shows that the definitions are equivalent in the locally convex linear topological space obtained by replacing the norm topology of a Banach space with the weak topology. [3,7].

2. *Introduction.* The definitions of compactness in a topological space are given below.

Definition A. A set in a topological space is *compact* if every open covering has a finite subcovering.

Definition A′. A set in a topological space is *relatively compact* if its closure is compact.

Definition B (B′). A set in a topological space is *semi-compact (relatively semi-compact)* if every infinite subset has a limit point in the set (a limit point in the space).

Definition C (C′). A set in a topological space is *sequentially compact (relatively sequentially compact)* if every sequence in the set has a subsequence which converges to a point in the set (to a point in the space).

The definitions of a sequentially compact set implies the definition for a semi-compact set and the similar implication for the relative definitions. In a Hausdorff space compact sets are semi-compact. There are examples of a sequentially compact topological space which is not compact, a compact topological space which is not sequentially compact, and a Hausdorff space which is not semi-compact but has a sequentially compact dense subset (see Grothendieck [5].

The three definitions for compact sets all imply their corresponding relative definitions, but the closure of a relatively semi-compact set or relatively sequentially compact set is not necessarily semi-compact or sequentially compact respectively. In a metric space the three definitions for compact sets are equivalent as are the corresponding relative definitions. In general the weak topology on a normed linear space is not metrizable or complete, thus leaving in doubt the relationships among the various definitions.

This paper establishes the fact that in the locally convex linear topological space obtained by putting the weak topology on a Banach

space, a normed linear complete space, the three definitions of compactness are equivalent and the closures of any of the three types of relatively compact sets are compact. The completeness hypothesis on the original Banach space is used only in showing that all semi-compact sets are compact.

3. *General Comments.* The linear space under discussion may be considered to have either the real or complex number for its scalar field. Theorem 2 which says that the relatively semi-compact sets are relatively sequentially compact in the weak topology of a normed linear space, and theorem 3 which says that in the weak topology of a Banach space, normed linear complete space, a relatively semi-compact set is relatively compact give the afore mentioned results. Theorem 2 is credited to Smulian [12, 13]. It also appears for real Banach spaces in a paper by Phillips [11]. Dieudonne and Schwartz [4] have proved the theorem for a locally convex, metrizable and complete linear space which they call an F space and have also proved the theorem for their LF spaces. This is not the F space mentioned by Banach [3] because of the additional postulate of local convexity which is imposed by Dieudonne and Schwartz. Further generalizations to function spaces and locally convex linear spaces have been given by Grothendieck [6].

Theorem 3 is an extension of Eberlein's theorem [5] for Banach spaces. The proof of theorem 3 in this paper eliminates the necessity of using Krein's theorem [9] on the weak compactness of convex hulls in conjunction with Eberlein's theorem to obtain the results of theorem 3. Dieudonne and Schwartz have proved theorem 3 for their F and LF spaces. A form of Eberlein's theorem is also given by Grothendieck [6].

4. *Notation.* X is a normed linear space having the real or complex numbers for its scalar field. An element belonging to X is denoted by x. X^* is the normed linear space of linear functionals on X, and element of X^* being x^*. In the same manner X^{**} in the normed linear space of functionals of X^*, x^{**} being an element of X^{**}.

The weak topology [3, 7] on X is generated by the elements of X^*. Take a finite set $(x^*_1, x^*_2, \ldots, x^*_n)$ of elements of X^*. The set of elements of X which have the property that $|x^*_i(x)| < 1$ for $i = 1,2,\ldots,n$ form a fundamental open neighborhood of the origin for the weak topology on X. Symbolically this neighborhood is written $U(\theta) = \{x\epsilon X \,|\, |x^*_i(x)| < 1,\ i = 1,2,\ldots\ldots,n\}$.

X generates a topology on X^* in the same manner as is used in obtaining the weak topology on X. This topology is called the

weak* topology of X^*. X^{**} also has a topology generated by X^* (the weak* topology on X^{**}) and in turn generates a topology on X^* (the weak topology on X^*).

5. *Relatively Semi-Compact: Relatively Seqentially Compact.* The definitions for sequential compactness and relative sequen-compactness may appear in a different form in some papers due to the fact that convergence of a sequence $\{x_n\}$ to a limit point x_o in the weak topology on X is equivalent to saying that the $\lim_{n\to\infty} x^*(x_n) = x^*(x_o)$ for all $x^*\epsilon X^*$.

The lack of the first countability axiom in the weak topology on X is the basis for the uncertainty of the relationship of semi-compactness to sequential compactness. In approaching this problem the following lemma is obtained for bounded sets in X^*. Let S^* be the unit ball in X^*.

LEMMA 1. If X is separable, the family of sets $\{(U = \{x^*\epsilon X^* | \; |x^*(x_i)| < \epsilon, \; i = 1,2,\ldots,n\}) \cap S^*\}$ in X^* have a countable base at the origin.

PROOF. Let x be a fixed element of X and take $U = \{x^*\epsilon X^* | \; |x^*(x)| < \epsilon, \; x\epsilon X\}$. Let N be a rational real number $< \epsilon$. Let m be an integer such that $N - \frac{1}{m} > 0$. There exists an x_k belonging to the set dense in X such that $||x_k - x|| < \frac{1}{m}$.

Let $U_k = \{x^*\epsilon X^* | \; |x^*(x_k)| < N - \frac{1}{m}\}$. It must now be shown that the $U_k \cap S^* \subset U \cap S^*$.
Let $x^*\epsilon U_k \cap S^*$.

$$\begin{aligned} |x^*(x)| &= |x^*(x) - x^*(x_k) + x^*(x_k)| \\ &\leq ||x^*|| \; ||x - x_k|| + |x^*(x_k)| \\ &\leq \frac{1}{m} + N - \frac{1}{m} = N < \epsilon \\ &\therefore x^*\epsilon U \cap S^*. \end{aligned}$$

For the general form $U = \{x^*\epsilon X^* | \; |x^*(x_i)| < \epsilon, \; i = 1,2,\ldots,n\}$, there exists an x_{i_k} for each x_i (the x_{i_k} belongs to the set dense in X) such that $||x_{i_k} - x_i|| < \frac{1}{m}$. Continue as before.

The number of sets of the form

$$U = \{x^*\epsilon X^* | \; |x^*(x_{i_k})| < M, \; i = 1,2, \ldots n\}$$

where M is a rational number, x_{i_k} belonging to a set dense in X, is countable.

Q. E. D.

In the above lemma S^* may be replaced (with proper changes) by any set of the form $\{x^*\epsilon X^* \mid \|x^*\| < c\}$, c any positive real number.

The above lemma now makes it possible to prove the following very useful theorem.

THEOREM 1. If the normed linear space X is separable then the first adjoint space X^* is separable in the weak* topology [3].

PROOF. It is sufficient to show that S^* is separable in the weak* topology.

Let $\mathcal{U} = \Big[U_n = \{x^*\epsilon X^* \mid |x^*(x_i)| < N,\ i = 1,2, \ldots n\} \Big|$ N a rational number and the x 's belonging to the countable set dense in $X\Big]$.

$\mathcal{U}$ is a countable family of sets.

Let $p\epsilon X^*$. Let $U_n(p) = \{x^*\epsilon X^* \mid |p(x_i) - x^*(x_i)| < N, i = 1,2, \ldots n\} = p + U_n$.

Holding U_n fixed, the collection $\{U_n(p) \mid p\epsilon S^*\}$ forms an open covering of S^* in the weak* topology of X^*. Since S^* is compact in the weak* topology [1, 4] there is a finite sub-covering.

Let B_n = the p's of the finite covering.

Let $C = \bigcup_n B_n$. C is a countable set in S^*.

For every $x^*\epsilon S^*$ and every $U_n \epsilon \mathcal{U}$ there exists a $p_n \epsilon C$ such that

$$x^*\epsilon U_n(p_n) \text{ or}$$
$$\therefore p_n \epsilon U_n(x^*).$$

By lemma 1 it is seen that for every $U(x^*)$ there exists a $U_n(x^*)$ such that $U_n(x^*) \cap S^* \subset U(x^*) \cap S^*$.

$\therefore S^* \subset$ the closure of C in the weak* topology.
$\therefore S^*$ is separable.
$\therefore X^*$ is separable in the weak* topology.

Q. E. D.

LEMMA 2. If the set W is relatively semi-compact in the weak topology on X then for each sequence $\{x_n\} \subset W$ there exists an $x_o \in X$ such that for each element $x^*\epsilon X^*$ there is a corresponding subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_{i\to\infty} x^*(x_{n_i}) = x^*(x_o)$ [11].

PROOF. Since W is relatively semi-compact in the weak topology there exists an $x_o \in X$ such that every neighborhood of x_o contains an infinite number of x_n's.

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For the general form $U = \{x^*\epsilon X^* | |x^*(x_i)| < \epsilon, i = 1,2,\ldots,n\}$, there exists an x_{i_k} for each x_i (the x_{i_k} belongs to the set dense in X) such that $||x_{i_k} - x_i|| < \frac{1}{m}$. Continue as before.

The number of sets of the form

$$U = \{x^*\epsilon X^* | |x^*(x_{i_k})| < M, i = 1,2, \ldots n\}$$

where M is a rational number, x_{i_k} belonging to a set dense in X, is countable.

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$$\therefore p_n \epsilon U_n(x^*).$$

By lemma 1 it is seen that for every $U(x^*)$ there exists a $U_n(x^*)$ such that $U_n(x^*) \cap S^* \subset U(x^*) \cap S^*$.

$\therefore S^* \subset$ the closure of C in the weak* topology.

$\therefore S^*$ is separable.

$\therefore X^*$ is separable in the weak* topology.

Q. E. D.

LEMMA 2. If the set M is relatively semi-compact in the weak topology on X then for each sequence $\{x_n\} \subset M$ there exists an $x_o \in X$ such that for each element $x^* \epsilon X^*$ there is a corresponding subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_{i \to \infty} x^*(x_{n_i}) = x^*(x_o)$ [11].

PROOF. Since M is relatively semi-compact in the weak topology there exists an $x_o \in X$ such that every neighborhood of x_o contains an infinite number of x_n's.

Let x^* be an arbitrary element of X^*. Hold x^* fixed. Let

$$S_0 = \{x_n\} = S'_1,$$

$$S_1 = \{x_n \in S'_1 \mid |x^*(x_n) - x^*(x_0)| < 1\}.$$

Take $x_{n_1} \in S_1$,

$$S'_2 = S_1 \cap \{S_1 \text{ - elts. } x_n \mid n \leqq n_1\},$$

$$S_2 = \{x_n \in S'_2 \mid |x^*(x_n) - x^*(x_0)| < \tfrac{1}{2}\}.$$

Take $x_{n_2} \in S_2$

$$S'_3 = S_2 \cap \{S_2 \text{ - elts. } x_n \mid n \leq n_2\}.$$

Continue this procedure.

The subsequence constructed above has the property that the

$$\lim_{i \to \infty} x^*(x_{n_i}) = x^*(x_0).$$

Since x^* was arbitraty, a subsequence with the desired property exixts for every $x^* \epsilon X^*$.

Q. E. D.

THEOREM 2. If the set M is relatively semi-compact in the weak topology of a normed linear space X then M is relatively sequentially compact in the weak topology.

PROOF. Let $\{x_k\}$ be a sequence in M. Let X_0 = smallest closed subspace of X containing $\{x_k\}$. Since X_0 is separable there exists a set of functionals, H, on X_0 dense in X^*_0 in the weak* topology (theorem 1).

Let $x \epsilon X_0$, $x \neq 0$. There exists an $x^* \epsilon X^*_0$ such that $x^*(x) \neq 0$. Let $|x^*(x)| = 2\epsilon$.
There exists an $\bar{x}^* \epsilon H$ such that $|x^*(x) - \bar{x}^*(x)| < \epsilon$.

$$\big|\, |x^*(x)| - |\bar{x}^*(x)| \,\big| < \epsilon$$

$$\big|\, 2\epsilon - |\bar{x}^*(x)| \,\big| < \epsilon$$

$$\therefore |\bar{x}^*(x)| > \epsilon > 0$$

$$\therefore \bar{x}^*(x) \neq 0, \ \bar{x}^* \epsilon H.$$

This shows that H is total on X.

Let K (use the Hahn-Banach-Bohnenblust-Sobczyk theorem on each element of H) be an extension of H to a set of functionals on X. Since H is total on X_0, K is total on X_0. By assuming a contradiction and making use of the uniform boundedness theorem it can be shown

that M is bounded in norm. Thus the real and imaginary parts of $x^*(x_k)$ are bounded for each $x^* \in X^*$. Using the diagonal procedure twice, once for the real part and once for the imaginary part of $x^*(x_k)$, we obtain a subsequence $\{x_n\}$ of $\{x_k\}$ such that the $\lim_{n\to\infty} x^*(x_n)$ exists for all $x^* \in K$. By lemma 2 there exists an $x_0 \in X$ such that for each $x^* \in X^*$ there is a corresponding subsequence $\{x_{n_i}\}$ such that the $\lim_{i\to\infty} x^*(x_{n_i}) = x^*(x_0)$.

We must now show that $x_0 \in X_0$. Assume $x_0 \notin X_0$, then by a corollary to the Hahn-Banach-Bohnenblust-Sobczyk extension theorem there exists an $x^* \in X^*$ such that $x^*(x_0) = 1$ and $x^*(x) = 0$ for all $x \in X_0$. This is a contradiction.

$$\therefore \quad x_0 \in X_0.$$

Suppose there exists an $x^*_0 \in X^*$ such that

$$\lim_{n\to\infty} x^*_0(x_n) \neq x^*_0(x_0).$$

Then there will exist an $\epsilon > 0$ and a subsequence $\{x_{n_j}\}$ such that

$$(*) \quad \lim_{j\to\infty} |x^*_0(x_{n_j}) - x^*_0(x_0)| \geqq \epsilon .$$

As before there exists an $x'_0 \in X_0$ such that for every $x^* \in X^*$ there is a corresponding subsequence $\{x_{n_{j_m}}\}$ such that

$$\lim_{n\to\infty} x^*(x_{n_{j_m}}) = x^*(x'_0).$$

But $x^*(x_0) = x^*(x'_0)$ for all $x^* \in K$,

$$\therefore \quad x_0 = x'_0 \quad (K \text{ total on } X_0).$$

This is a contradiction to $(*)$.

$$\therefore \quad \lim_{n\to\infty} x^*(x_n) = x^*(x_0) \text{ for all } x^* \in X^*.$$

Q. E. D.

6. *Relatively Semi-Compact: Compact.*

LEMMA 3. Given an $x^{**} \in X^{**}$, x^{**} is in X if and only if $Q_{x^{**}} = \{x^* \in X^* \mid x^{**}(x^*) = 0\}$ is closed in the weak* topology of X^*.

The classical reference for the proof of the above lemma is pages 131 and 132 of Banach's book [3]. A second look at the lemma shows that is states a necessary and sufficient condition for x^{**} to be a continuous linear functional on the locally convex linear topological space obtained by putting the weak* topology on X^*. Since X is the totality of linear functionals on this space [2], x^{**} must correspond to an element of X.

While lemma 3 is true for locally convex linear topological spaces, the new version given in lemma 4 requires some type of completeness on the space. In this paper it is assumed that the original space is a Banach space. The completeness makes it possible to assume that $Q_{x^{**}}$ is closed in the weak* topology if its intersection with the set $S^* = \{x^* \in X^* \mid \|x^*\| \leqq 1\}$ is closed in weak* topology. This is proved in a slightly different form on pages 118 and 121 of Banach's book [3].

LEMMA 4. A necessary and sufficient condition that an element $x^{**} \in X^{**}$ be in the Banach space X is that there exists in X a set M, relatively semi-compact in the weak topology, with the following property:
Given an arbitrary set $(x^*_1, x^*_2, \ldots, x^*_n) \subset X^*$, there exists an x belonging to the closure of M in the weak topology such that $x^{**}(x^*_i) = x^*_i(x)$, $(i = 1, 2, \ldots, n)$ [5].

PROOF. The necessity is obtained directly from the definitions.

Sufficiency: The lemma is reduced to showing:
If $g \in X^*$ is a limit point of $Q_{x^{**}} \cap S^*$ in the seak* topology of X^* then $x^{**}(g) = 0$ (see lemma 3 and discussion).

Construct three sequences.

$\{x_n\} \subset M \subset X$

$\{y_n\} \subset \overline{M}$ (closure of M in the weak topology) $\subset X$

$\{x^*_n\} \subset Q_{x^{**}} \cap S^* \subset X^*$

Proceed as follows:
Given an arbitrary $\epsilon > 0$,
there exists a $y_1 \in \overline{M}$ such that $g(y_1) = x^{**}(g)$ (by hyp.)
there exists an $x_1 \in M$ such that $|g(y_1) - g(x_1)| < \frac{\epsilon}{4}$,
there exists an $x^*_1 \in Q_{x^{**}} \cap S^*$ such that $|g(x_1) - x^*_1(x_1)| < \frac{\epsilon}{4}$
(because g is a weak* limit point.)
there exists a $y_2 \in \overline{M}$ such that $g(y_2) = x^{**}(g)$ (by hyp.)

$$x^*_1(y_2) = x^{**}(x^*_1) = 0,$$

there exists an $x_2 \in M$ such that

$$|g(y_2) - g(x_2)| < \frac{\epsilon}{4}$$

$$|x^*_1(y_2) - x^*_1(x_2)| < \frac{\epsilon}{4}$$

there exists $x^*_2 \in Q_{x^{**}} \cap S^*$ such that

$$|g(x_1) - x^*_2(x_1)| < \frac{\epsilon}{4}$$

$$|g(x_2) - x^*_2(x_2)| < \frac{\epsilon}{4}$$

and so on.

The above sequences have the following properties:

a) $||x^*_m|| \leqq 1 \quad (m = 1,2, \ldots ,)$

b) $x^*_m(y_n) = 0 \quad (m = 1,2, \ldots , n - 1)$

c) $|x^{**}(g) - g(x_n)| < \frac{\epsilon}{4} \quad (n = 1,2, \ldots ,)$

d) $|x^*_m(y_n) - x^*_m(x_n)| < \frac{\epsilon}{4} \quad (m = 1,2, \ldots , n - 1)$

e) $|g(x_n) - x^*_m(x_n)| < \frac{\epsilon}{4} \quad (n = 1,2, \ldots , m)$.

From c and e it follows that

f) $|x^{**}(g) - x^*_m(x_n)| < \frac{\epsilon}{2} \quad (n = 1,2, \ldots , m)$.

Since M is relatively sequentially compact (theorem 2) $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ such that the $\lim_{i\to\infty} x^*(x_{n_i}) = x^*(x)$ for all $x^* \in X^*$. x belongs to $\overline{M}$ because the sequential closure is contained in the weak closure.

Notation: $\{x_{n_i}\}$ will now be written $\{x_n\}$. The corresponding $\{y_{n_i}\}$ and $\{x^*_{m_i}\}$ will now be $\{y_n\}$ and $\{x^*_m\}$.

From b and d it follows that

$$|x^*_m(x_n)| < \frac{\epsilon}{4} \quad (m = 1,2, \ldots , n - 1)$$

$$\lim_{n\to\infty} |x^*_m(x_n)| \leq \frac{\epsilon}{4} \quad \text{for all } m.$$

$$\therefore |x^*_m(x)| \leq \frac{\epsilon}{4} \quad \text{for all } m.$$

Let X be considered as a real linear space X_r. $\{x_n\}$ converges to x in $\overline{M} \subset X$, i. e. $x^*(x_n) \to x^*(x)$ for every $x^* \in X^*$. This implies $x^*_1(x_n) \to x^*_1(x)$ for every $x^*_1 \in (X_r)^* \cong (X^*)_r$, hence $\{x_n\}$ converges weakly to x in $\overline{M} \subset X_r$. By Mazur's theorem [10] on the weak and norm closure of a convex set, there exists a $z \in X_r$ such that $||x-z|| < \frac{\epsilon}{4}$, where

$$z = \sum_{i=1}^{k} a_i x_{n_i}, \text{ with } \sum_{i=1}^{k} a_i = 1,\ a_i \geq 0.$$

Put $m = n_k$ in (f). By adding the k inequalities (f) for $n = n_1, n_2, \ldots, n_k$ and using the triangle inequality the following inequality is obtained.

$$|x^{**}(g) - x^*_{n_k}(z)| \leq \sum_{i=1}^{k} a_i |x^{**}(g) - x^*_{n_k}(x_{n_i})| < \frac{\epsilon}{2},$$

$$|x^{**}(g)| = |x^{**}(g) - x^*_{n_k}(z) + x^*_{n_k}(z) - x^*_{n_k}(x) + x^*_{n_k}(x)|$$

$$\leq |x^{**}(g) - x^*_{n_k}(z)| + |x^*_{n_k}(z) - x^*_{n_k}(x)| + |x^*_{n_k}(x)|$$

$$< \frac{\epsilon}{2} + ||x^*_{n_k}||\ ||z-x|| + \frac{\epsilon}{4} < \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon$$

Hence, $|x^{**}(g)| < \epsilon$ for every $\epsilon > 0$ which implies $x^{**}(g) = 0$ as required.

THEOREM 3. A relatively semi-compact set M in the weak topology of Banach space X is relatively compact in the weak topology.

PROOF. Let $\overline{M}$ be the closure of M in the weak topology. Consider M as a set in X^{**}. M is bounded in norm in X^{**} [4, 8]. The theorem is finished if it is shown that $\overline{M}$ is closed in the weak* topology of X^{**}.

Let x^{**} be a limit point of M in the weak* topology of X^{**}. Let $(x^*_1, x^*_2, \ldots, x^*_n)$ be an arbitrary set of X^*. There exists a sequence $\{x_m\} \subset M$ such that $|x^{**}(x^*_i) - x^*_i(x_m)| < \frac{1}{m}$, $i = 1, 2, \ldots, n$. Since M is sequentially compact (theorem 2) there exists a subsequence $\{x_{m_j}\}$ and an $x \in M$ such that $\lim_{i\to\infty} x^*(x_{m_j}) = x^*(x)$ for all $x^* \in X^*$.

$$|x^{**}(x^*_i) - x^*_i(x)| \leqq x^{**}(x^*_i) - x^*_i(x_{m_i})|$$
$$+ |x^*_i(x_{m_j}) - x^*_i(x)|, \quad (i = 1, 2, \ldots, n).$$

Let $m_i \to \infty$

$\therefore x^{**}(x^*_i) = x_i\ (x) \qquad (i = 1, 2, \ldots, n)$

$\therefore x^{**} \in X$ (lemma 5)

$\therefore x^{**} \in \overline{M}$

$\therefore \overline{M}$ is closed in the weak* topology of X^{**} and bounded in norm.

By Alaoglu's theorem it is observed that $\overline{M}$ is compact in X^{**}. Since X with the weak topology has the subset topology of X considered as imbedded in X^{**} (with the weak* topology) and since $\overline{M} \subset X$, it is concluded that $\overline{M}$ is compact for the weak topology on X.

Q. E. D.

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ON BARBIER'S SOLUTION OF THE BUFFON NEEDLE PROBLEM

Murray S. Klamkin

In the well known Buffon needle problem, a needle of length L is dropped on a board ruled with equidistant parallel lines of spacing D where $D \geq L$; it is required to determine the probability that the needle will intersect one of the lines.

Barbier's [1] elegant method was to let this problem depend on the following one: What is the mathematical expectation of the number of points of intersection when a polygonal line (convex or not) is thrown upon the board? The perimeter of the polygonal line can be subdivided into N rectilinear parts $a_1, a_2, \cdots, a_N$, all less than D. Associated with these N parts are the variables $x_1, x_2, \cdots, x_N$, such that

$$x_r = 1, \quad \text{if one of the board lines intersects } a_r,$$

$$x_r = 0, \quad \text{otherwise.}$$

The sum $T = \sum_1^N x_r$ gives the total number of points of intersection of the polygonal line with the board lines. Hence, $E(T) = \sum_1^N E(x_r)$, where $E(x_r)$ is the mathematical expectation of x_r. If P_r is the probability of intersection of a_r with one (and only one) line, then $E(x_r) = P_r$. If it is imagined that the needle is divided into two parts L_1 and L_2 then evidently a line intersects the needle, if and only if, it intersects either the first or second part. Hence, if $P(L)$ denotes the probability of intersection for a needle of length L, then

$$P(L_1 + L_2) = P(L_1) + P(L_2), \tag{1}$$

by the theorem of total probability. It then follows that $P(L) = CL$, where C is a constant independent of L. Thus, $E(T) = C\sum_1^N a_r = CS$, where S is the perimeter of the polygonal line. This latter result is also valid for any curvilinear arc (closed or not) as can be seen by the method of limits. Since a circle of diameter D always has exactly two points of intersection with the ruled lines of the board, it follows immediately that $2 = C\pi D$. Consequently,

$$P(L) = 2L/\pi D. \tag{2}$$

There are two objections to this ingenious method. The first one is to the tacit assumptions which lead to Eq. (1). The second one is to the assumption that the expectation $E(T)=CS$, is independent of the shape of the polygonal line. These assumptions imply some condition on the distribution function. It should be noted here, however, that the second assumption includes the first one. For if we imagine the needle bent back on itself N times such that its length is L/N, we must then have $E=1\times P(L)=N\times P(L/N)$.

If the needle position is determined by the distance x, of its middle point from the nearest line, and by the acute angle, ϕ, between this perpendicular distance and the needle, the probability of intersection is given by

$$(3) \qquad P(L) = \frac{\int_0^{\pi/2}\int_0^{\frac{L}{2}\cos\phi} F(x,\phi)\,dx\,d\phi}{\int_0^{\pi/2}\int_0^{D/2} F(x,\phi)\,dx\,d\phi},$$

where $F(x,\phi)$ is the distribution function in the variables x and ϕ. Consequently, Eq. (1) implies the following condition on $F(x,\phi)$:

$$(4) \qquad \int_0^{\pi/2}\int_0^{\frac{L_1+L_2}{2}\cos\phi} F(x,\phi)\,dx\,d\phi = \int_0^{\pi/2}\int_0^{\frac{L_1}{2}\cos\phi} F(x,\phi)\,dx\,d\phi + \int_0^{\pi/2}\int_0^{\frac{L_2}{2}\cos\phi} F(x,\phi)\,dx\,d\phi.$$

If we assume that $F(x,\phi)$ has a power series expansion[1]

$$(5) \qquad F(x,\phi) = \sum_{r=0}^{\infty} a_r\, x^r(\phi),$$

then Eq. (4) implies that $F(x,\phi)=F(\phi)$. Consequently, Eq. (3) reduces to

$$(6) \qquad P(L) = \frac{L}{D}\,\frac{\int_0^{\pi/2} F(\phi)\cos\phi\,d\phi}{\int_0^{\pi/2} F(\phi)\,d\phi}.$$

We will now derive the condition implied on $F(x,\phi)$ by assuming that the expectation $E(T)$ is independent of the shape of the polygonal line. Let us consider a needle bent at its midpoint M to form an open triangle AMB, where

(7) $AM = MB = a, \quad AB = L, \quad \measuredangle AMB = 2\theta,$ median to side $AB = m$.

[1] This is equivalent to assuming that x, and ϕ, are independent variables, and that $F(x,\phi)$ is regular in x.

Also, let the position of AB on the board be specified by the variables x and ϕ (defined as before). Then the bent needle can have four different positions on the board as shown in Figure 1.

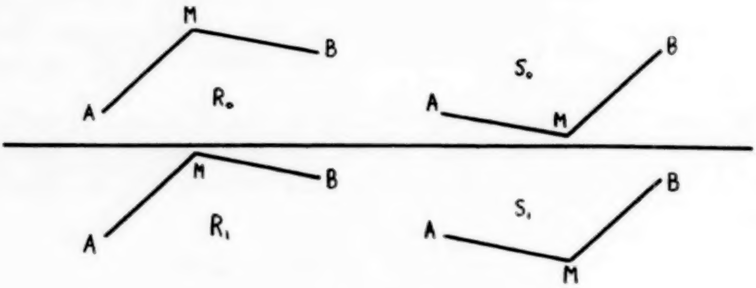

Figure 1.

R_0 and S_1 represent equivalent configurations, and similarly, S_0 and R_1. Consequently, we can ignore configurations of the type R_1 and S_1. The total phase space ("domain" as used by Uspensky) spanned by the configuration R_0 (M above AB) is

$$0 \le \phi \le \pi/2, \qquad (8)$$
$$0 \le x \le D/2.$$

This is the same for the configuration S_0 (M below AB). The following table gives the portion of the phase space spanned for 0, 1, or 2 intersections by the bent needle AMB with the ruled lines for configurations R_0 and S_0:

Configuration

Number of Intersections	R_0	S_0	
0	$0 \le \phi \le \frac{\pi}{2}$ $\frac{L}{2}\cos\phi \le x \le \frac{D}{2}$	$0 \le \phi \le \theta$ $\frac{L}{2}\cos\phi \le x \le \frac{D}{2}$	$\theta < \phi \le \frac{\pi}{2}$ $m\sin\phi < x \le \frac{D}{2}$
1	$0 \le \phi \le \frac{\pi}{2}$ $0 \le x \le \frac{L}{2}\cos\phi$	$0 \le \phi \le \theta$ $0 \le x \le \frac{L}{2}\cos\phi$	$\theta < \phi \le \frac{\pi}{2}$ $0 < x \le \frac{L}{2}\cos\phi$
2			$\theta \le \phi \le \frac{\pi}{2}$ $\frac{L}{2}\cos\phi \le x \le m\sin\phi$

As a check, we find that the sum of the portions of the phase space

exhibited in the table adds up to the total phase space spanned by the configurations R_0 and S_0.

It follows from the table that the mathematical expectation for the number of points of intersection of the bent needle is given by

$$(9)\qquad E = \frac{2\int_0^{\pi/2}\int_0^{\frac{L}{2}\cos\phi} F(x,\phi)\,dx\,d\phi + 2\int_\theta^{\pi/2}\int_{\frac{L}{2}\cos\phi}^{m\sin\phi} F(x,\phi)\,dx\,d\phi}{2\int_0^{\pi/2}\int_0^{D/2} F(x,\phi)\,dx\,d\phi}$$

If E is to be independent of θ, we must have,

$$(10)\qquad \frac{\int_0^{\pi/2}\int_0^{a\sin\theta\cos\phi} F(x,\phi)\,dx\,d\phi + \int_\theta^{\pi/2}\int_{a\sin\theta\cos\phi}^{a\cos\theta\sin\phi} F(x,\phi)\,dx\,d\phi}{\int_0^{\pi/2}\int_0^{D/2} F(x\ \phi)\,dx\,d\phi} = \frac{4a}{\pi D},$$

since $L = 2a\sin\theta$, and $m = a\cos\theta$.

Since we have shown previously that $F(x,\phi) = F(\phi)$ [from Eq. (5)], Eq. (10) reduces to

$$(11)\qquad \sin\theta\int_0^\theta F(\phi)\cos\phi\,d\phi + \cos\theta\int_\theta^{\pi/2} F(\phi)\sin\phi\,d\phi = k_1 \quad \text{(constant)}.$$

Differentiating Eq. (11) twice with respect to θ yields

$$(12)\qquad -\sin\theta\int_0^\theta F(\phi)\cos\phi\,d\phi - \cos\theta\int_\theta^{\pi/2} F(\phi)\sin\phi\,d\phi + F(\theta) = 0.$$

Consequently, $F(\theta)$ = constant.

Thus it has been demonstrated that the assumption that E is independent of the shape of the polygonal line is equivalent to the assumption that the variables x and ϕ are uniformly distributed. This being the case, we can use Eq. (3) directly to obtain the probability of intersection $P(L)$.

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A TYPE OF PERIODICITY FOR FIBONACCI NUMBERS

Vern Hoggatt

The question was raised as to the existence of three Fibonacci* numbers which are squares of sides of a right triangle. It is easily deduced that, if so, the Fibonacci numbers must be consecutive. The triangle must then be a Pythagorean primitive because no two consecutive Fibonacci numbers have a common factor greater than 1. A possible attack on this problem lies in a type of periodicity of the Fibonacci numbers, which is discussed below.

We define $T_k(A)$ as the whole number whose k digits are the terminal k digits of an integer $A \geq 0$, i.e. $T_2(124) = 24$. If A does not have k digits as ordinarily written, consider the left-most spaces necessarily blocked in with zeroes. The symbol $T_k(A)$ has the following properties:

(1) $$T_k(A) \geq 0 \quad \text{for} \quad A \geq 0$$

(2) $$T_k(A+B) = T_k(T_k(A) + T_k(B)) \qquad A, B \geq 0$$

e.g. $$T_2(330) = T_2(164+166) = T_2(64+66) = T_2(130) = 30$$

(3) $$T_k(A-B) = T_k(10^k + T_k(A) - T_k(B)) \qquad A \geq B \geq 0$$

e.g. $$T_2(272-173) = T_2(100+72-73) = T_2(99) = 99$$

(4) $$T_k(A \times B) = T_k(T_k(A) \times T_k(B)) \qquad A, B \geq 0$$

e.g. $T_2(32580) = T_2(180 \times 181) = T_2(80 \times 81) = T_2(6480) = 80$. If

(5) $T_k(A) = T_k(B)$ then $T_s(A) = T_s(B)$ for $s = 1, 2, 3, \cdots, k$.

*The Fibonacci numbers are given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$.

Using property (2) with the recursion formula $F_{n+2} = F_{n+1} + F_n \quad n \geq 0$

$$T_k(F_{n+2}) = T_k[T_k(F_{n+1}) + T_k(F_n)]$$

the first 60 of the $T_1(F_n)$, noting that $F_0 = 0$ and $F_1 = 1$, can be readily written down: 0,1,1,2,3,5,8,3,1,4,5,9,4,3,7,0,7,7,4,1,5,6,1,7,8,5,3, 8,1,9,0,9,9,8,7,5,2,7,9,6,5,1,6,7,3,0,3,3,6,9,5,4,9,3,2,5,7,2,9,1,0, 1, $\cdots$. Thus $T_1(F_{n+60}) = T_1(F_n)$ for $n \geq 0$. Since 60 is the smallest integer for which the foregoing holds for all nonnegative integral n, we call 60 the period of $T_1(F_n)$. By direct calculation we find for $T_2(F_n)$ a period of 300, and for $T_3(F_n)$ a period of 1500. These calculations were not necessarily extensive* since, obviously, the period of $T_k(F_n)$ is divisible by the period of $T_s(F_n)$ for all integral s less than or equaling k. Hence we are led to define integers p_k to be the smallest positive integer for which $T_k(F_{n+p_k}) = T_k(F_n)$ for all $n \geq 0$. We will now prove the existence of such p_k and in fact will show:

Theorem: $p_k = 1.5 \times (10)^k$ *for* $k \geq 3$.

Proof: We first write down a well-known formula concerning Fibonacci numbers.

A. $\qquad F_{n+p+1} = F_{n+1}F_{p+1} + F_nF_p \quad$ for $n, p \geq 0$

In computing several $T_{10}(F_n)$ on a Monroe Calculator the following lemma was observed:

Lemma I: If $F_{M-1} = A \cdot 10^s + 1$ *and* $F_M = B \cdot 10^s$ *for positive integral* A *and* B, *then*

$$T_{2s-1}(F_{mM-1}) = T_{2s-1}(m[F_{M-1} - 1] + 1) \quad \textit{and}$$

$$T_{2s-1}(F_{mM}) = T_{2s-1}(mF_M)$$

for $m = 1, 2, 3, \cdots$.

The proof of the lemma will proceed by induction. The lemma is obviously true for $m = 1$. The assumption that the lemma is true for some $m = k$ allows us to write down the following: Let $T_{2s-1} = T$ for simplicity,

a. $\qquad T(F_{M-1}) = T(A \cdot 10^s + 1)$

b. $\qquad T(F_M) = T(B \cdot 10^s)$

*There is no need to compute all the $T_2(F_n)$ for $n = 1, 2, \cdots, 300$. It suffices to compute using formula A below and properties (2) and (4) above $T_2(F_{59})$ and $T_2(F_{60})$; $T_2(F_{119})$ and $T_2(F_{120})$; $T_2(F_{179})$ and $T_2(F_{180})$; $T_2(F_{239})$ and $T_2(F_{240})$; and finally $T_2(F_{299})$ and $T_2(F_{300})$. For the next period we started with $T_3(F_{59})$ and $T_3(F_{60})$ and repeated till we got $T_3(F_{299})$ and $T_3(F_{300})$ and then computed $T_3(F_{r(300)-1})$ and $T_3(F_{r(300)})$ for $r = 1, 2, \cdots, 5$.

c. $$T(F_{M+1}) = T((A+B)\cdot 10^s + 1)$$

d. $$T(F_{kM-1}) = T(k[F_{M-1} - 1] + 1)$$

e. $$T(F_{kM}) = T(kF_M)$$

Equivalents a, b, c, are from the hypotheses of the lemma while d and e rest on the validity of the conclusion of the lemma for $m = k$. Using formula A with $n = kM - 1$, $p = M$ and properties (2) and (4) we may write

$$T(F_{[k+1]M}) = T\{T[T(F_{kM}) \times T(F_{M+1})] + T[T(F_{kM-1}) \times T(F_M)]\}.$$

Substituting for each left side of b, c, d, e, its equivalent into the above expression and applying properties (4) then (2) from right to left we obtain

$$T(F_{[k+1]M}) = T\{k[2AB + B^2]\cdot 10^{2s} + [k+1]B\cdot 10^s\}.$$

Since $T_{2s-1}(k[2AB+B^2]\cdot 10^{2s}) = 0\cdots 0$, we easily obtain our first result

R_1: $$T_{2s-1}(F_{[k+1]M}) = T_{2s-1}([k+1]B\cdot 10^s) = T_{2s-1}([k+1]F_M).$$

Next, again using formula A with $n = kM - 1$, $p = M - 1$ and properties (2) and (4) left to right, we obtain

$$T(F_{[k+1]M-1}) = T\{T[T(F_{kM}) \times T(F_M)] + T[T(F_{kM-1}) \times T(F_{M-1})]\}.$$

Substituting for each left side of a, b, d, e its equivalent into the above expression and applying properties (4) and (2) from right to left we obtain

$$T(F_{[k+1]M-1}) = T\{k[A^2 + B^2]\cdot 10^{2s} + [k+1]A\cdot 10^s + 1\}.$$

Since $T_{2s-1}(k[A^2+B^2]\cdot 10^{2s}) = 0\cdots 0$, we easily obtain our second result

R_2: $$T_{2s-1}(F_{[k+1]M-1}) = T_{2s-1}([k+1]A\cdot 10^s + 1) = T_{2s-1}([k+1](F_{M-1} - 1) + 1).$$

Results R_1 and R_2 together is the conclusion of the lemma for $m = k+1$. Since the conclusion of the lemma is true for $m = 1$, the induction is complete.

We note here that, if the hypotheses of the lemma are satisfied, (since $F_0 = 0\cdots 0$ and $F_1 = 0\cdots 01$), $T_s(F_M) = T_s(F_0)$ and $T_s(F_{M+1}) = T_s(F_1)$. One then concludes that $p_s \le M$, since $T_s(F_{M+n}) = T_s(F_n)$ for all $n \ge 0$. These hypotheses are satisfied for $s = 3$, $M = 1500$, and the existence of p_s is guaranteed for $s, s+1, \cdots, 2s-1$ by letting $m = 1, 10, 100, \cdots, 10^{s-1}$. We

then choose a new $q = 2s - 1$, $M' = p'_s\ 10^{s-1}$ and proceed. By continuing we thus show existence of all p_s for $s = 3, 4, 5, \cdots$.

Lemma II: If $F_{p_s - 1} = A \cdot 10^s + 1$, $F_{p_s} = B \cdot 10^s$, where A is an odd positive integer and if $s \geq 3$ then $F_{10p_s - 1} = A' \cdot 10^{s+1} + 1$, $F_{10p_s} = B' \cdot 10^{s+1}$ with A' an odd positive integer. Further $p_{s+1} = 10p_s$, if $A \not\equiv$ mod 5.

The above hypotheses include those of Lemma I, therefore we may write with $M = p_s$, $s \geq 3$, $m = 10$ using property (5) with results of Lemma I

$$T_{s+1}(F_{10M-1}) = T_{s+1}(10[F_{M-1} - 1] + 1) \quad \text{and} \quad T_{s+1}(F_{10M}) = T_{s+1}(10F_M).$$

From which it follows that

$$F_{10M-1} = A' \cdot 10^{s+1} + 1 \quad \text{and} \quad F_{10M} = B' \cdot 10^{s+1}$$

for A' and B' positive integers. But since $F_{10M-1} - 1$ ends in $(s+1)$ zeros and $T_{s+2}(F_{10M-1}) = T_{s+2}(10(F_{M-1} - 1) + 1)$ it follows that A and A' have the same terminal digit hence A' is odd if A is odd.

Since $F_{10M-1} = A' \cdot 10^{s+1} + 1$ and $F_{10M} = B' \cdot 10^{s+1}$, it follows that $p_{s+1} \leq 10M$ and also p_{s+1} divides $10M = 10p_s$ and p_s divides p_{s+1} but obviously since A is odd $p_s < p_{s+1}$ hence $p_{s+1} = 2p_s$, $5p_s$ or $10p_s$. Since evidently $F_{(p_{s+1}-1)} - 1$ must end in $(s+1)$ zeros, it follows that $p_{s+1} \neq 2p_s$ or $5p_s$ hence $p_{s+1} = 10p_s$ for $s \geq 3$. This proves all of lemma II.

Hence if one can find some $F_{p_q} = B \cdot 10^q$ and $F_{p_q - 1} = A \cdot 10^q + 1$ with A odd then it would follow by induction that $p_{s+1} = 10p_s$ for $s \geq q \geq 3$.

By direct calculations these conditions are met for $s = 3$, $p_3 = 1500$, and A terminating in unity. Specifically: $T_4(F_{1499}) = 1001$ and $T_4(F_{1500}) = 8000$. It follows that $T_5(F_{14999}) = 10001$ and $T_5(F_{15000}) = 80000$ and further that $p_4 = 15{,}000$.

Therefore we complete the proof by induction that $p_{s+1} = 10p_s$ for $s \geq 3$ and that $p_k = (1.5) \times 10^k$ for $k \geq 3$.

Summary: The periods of $T_k(F_n)$ are $p_1 = 60$, $p_2 = 300$, $p_k = (1.5) \times 10^k$ for $k \geq 3$. Since the terminal k digits of F_n form a periodic sequence, it would seem possible to prove the nonexistence of three consecutive Fibonacci numbers which are squares, if one could show that a complete cycle was devoid of any such three consecutive numbers of the form $T_k(n^2)$ which are the terminal k digits of some square. The author has attempted unsuccessfully to do this up through the first 15,000 or so of the F_n. There is considerable labor involved on a hand computer but the computation of the terminal k digits of the F_n is nothing but a straightforward routine process. Using formula A with properties (2) and (4) helps make bigger jumps. Perhaps this could be investigated further by some automatic equipment. My conjecture is that no such Fibonacci numbers exist.

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ON THE GRAPHICAL SOLUTION OF CUBIC EQUATIONS

S. Kulik

1. Consider cubic equations of the type

(1) $$x^3 + px + q = 0.$$

With p and q as variables and x a parameter, this represents in Cartesian Coordinates, a family of straight lines. Each of the straight lines is a tangent to the envelope whose equation is

(2) $$(p/3)^3 + (q/2)^2 = 0.$$

This gives a simple method for the graphical solution of cubic equations of type (1) above[1].

Consider a given equation $x^3 + px + q = 0$. This fixes a point on the straight line whose coordinates are (p, q). Drawing the straight line tangential to the envelope (2), it will cut the axes of the coordinates, making segments m and n, (Fig. 1) equal in length to the square and cube of the root of the equation, on OP and OQ respectively, but with opposite signs.

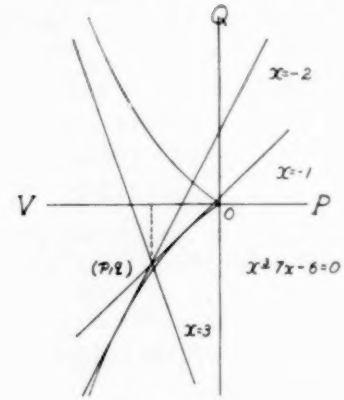


Figure 1.

Thus we can write

$$x^2 = -m, \quad x^3 = -n$$

hence

(3) $$x = \pm\sqrt{-m}, \quad x = -\sqrt[3]{-n}, \quad x = n/m$$

Each of the three representations should give the same value. By comparing the square and cube roots it is possible to get a root of the equation very accurately.

1 Compare: Runge, C., *Graphical Methods*, N, Y., Columbia Univ. Press, 1912, p. 59.

If the point (p,q) is situated to the left of the envelope (2), there are three tangents from this point to the envelope. In this case

$$(p/3)^3 + (q/2)^2 < 0.$$

The equation has three real roots; two positive and one negative if the point is above axis OP; two negative and one positive if the point is below it.

If the point is situated on the envelope itself, then

$$(p/3)^3 + (q/2)^2 = 0.$$

The equation has three real roots; two of them are equal. There will be only two tangents to the envelope through the point (p,q). The tangent at the point (p,q) corresponds to the equal roots.

If the point is situated to the right of the envelope, then

$$(p/3)^3 + (q/2)^2 > 0.$$

The equation has only one real root and there will be only one tangent to the envelope.

2. An alternative method is obtained by using the involute of the envelope,

$$q^2 + 4p - 8 = 0 \qquad (4)$$

which is a parabola, (Fig. 2).

A normal to this parabola drawn through the point (p,q) where p and q are the coefficients of the equation $x^3 + px + q = 0$, has at the point of the normal intersection with the parabola, the following coordinates:

$$p_1 = 2 - x^2, \qquad q_1 = -2x$$

Hence

$$x = \pm\sqrt{2 - p_1}, \qquad x = -q_1/2. \qquad (5)$$

The sign of the square root should be chosen to correspond with the value of $-q_1/2$.

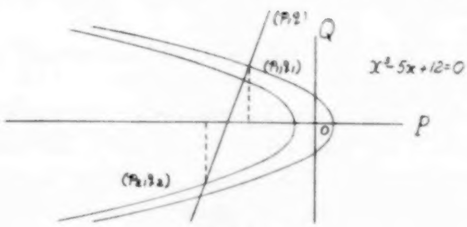


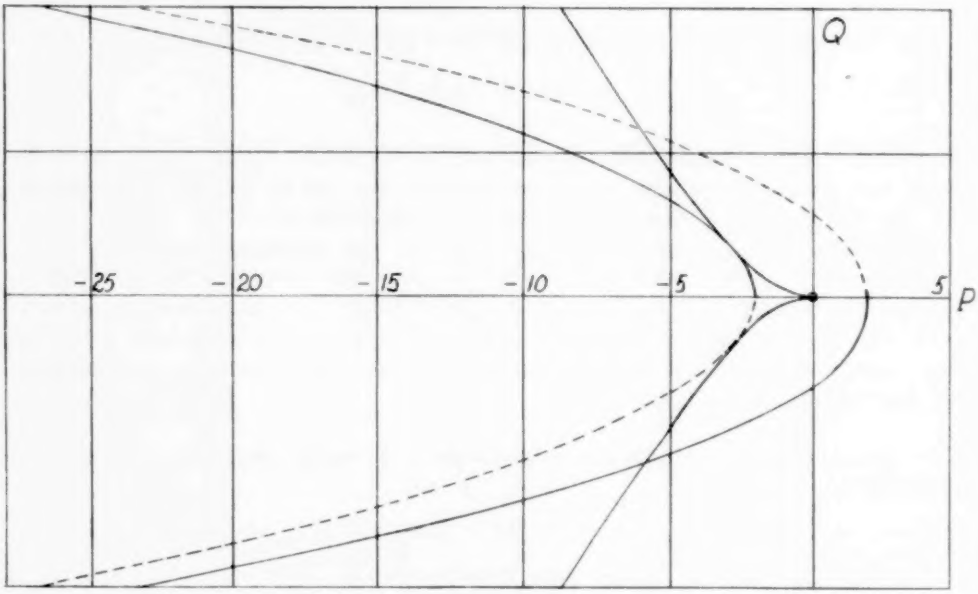
Figure 2.

3. A very attractive method of two parabolae will be apparent if we consider the parabola given by equation (4) together with that given by

the equation

(6) $$q^2 + 4p + 8 = 0$$

which is the same parabola shifted 4 units in the negative direction of OP, (Fig. 2), (Graph 1).



Graph 1.

The normal drawn to the first parabola through the point (p,q) would intersect the opposite branch of the second parabola at a point (p_2, q_2), where

$$p_2 = -2 - x^2, \qquad q_2 = 2x$$

and thus

(7) $$x = \pm\sqrt{-2-p_2}, \qquad x = q_2/2$$

The two points (p_1, q_1), (p_2, q_2) are situated symmetrically with respect to the point $(-x^2, 0)$ on the axis OP.

Therefore, *by putting a ruler through the point (p,q) on a graph of the two parabolae, (4) and (6), it is only necessary to ensure that the two ordinates q_1, q_2 are equal in absolute value. The edge of the ruler will also be normal to the outer parabola. Then $x = q_2/2$ is the root of the equation.*

4. The method of the two parabolae can be easily generalized. Let us consider a parabola represented by the equation

$$q^2 + k^2 p + k^3 = 0. \tag{8}$$

One point of intersection of this parabola with the straight line $x^3 + px + q = 0$ has coordinates $(-k-x^2, kx)$. Supposing k positive and negative in turn we shall get two parabolae cut by a straight line in two points symmetrically situated with respect to the point $(-x^2, 0)$.

The root of the equation is then

$$x = q_2/k \tag{9}$$

The straight line, however, is not, generally, normal to either of the parabolae.

University of South Carolina

AN INQUIRY

Mathematics Magazine
Pacoima, California

Sirs;- Will you let me know if there is a book explaining how the ancient Romans solved algebraic and geometrical problems with the Roman numerals.

I am a machinist by trade and have been told that the Romans used an abacus for arithmetical problems but can't find out how they solved other branches of mathematics with the Roman numerals.

Hoping you can furnish me with this information, I remain

Yours Respectfully,
(signed) J. F. Baxter
20195 Cardoni Street
Detroit 3, Michigan

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Alfred, New York.

ON INTRODUCING ARGUMENTS INTO FRESHMAN ALGEBRA

Gerald B. Huff

There seems to be a wide spread dissatisfaction with the conventional freshman algebra course as it has been given in American colleges. We teachers get complaints from deans, faculty colleagues, and the freshmen themselves. These people, particularly the latter, are concerned over the high mortality rate. In this talk, I will restrict myself to some common complaints of college mathematics teachers and suggest a possible remedy. None of us enjoy the tedium of teaching formal manipulation, which might have been learned earlier, to freshmen who have, in high school, learned only to dislike a subject called algebra. Such formal courses do not prepare the college student for the mathematical statements and arguments he may meet in mathematics courses. What is perhaps worse, the undergraduates who are induced to elect advanced mathematics courses may not be the group best able to benefit from them. Various colleges have attacked this problem in various ways. Text books have been written which provide suitable material for better freshman courses, and many "standard" texts reflect this trend.

My own approach has been to introduce "if - then" statements and arguments at every opportunity in doing the usual manipulation. This approach may be illustrated by the following examples. The manipulation carried out in "solving" the equation $3x + 2 = 7$ provides an argument which proves the statement: if there is a rational number x such that $3x + 2 = 7$, then $x = 5/3$. The elimination of y from $2x + 3y = 7$, $x + y = 3$ by any method shows that: if there is a pair of numbers x_0, y_0 satisfying both these equations, then $x_0 = 2$. A bit of manipulation in almost any book is the framework for showing that: if there is a pair of numbers x_0, y_0 satisfying both equations $ax + by = c$, $Ax + By = C$, then

$$\begin{vmatrix} a & b \\ A & B \end{vmatrix} x_0 = \begin{vmatrix} c & b \\ C & B \end{vmatrix} \text{ and } \begin{vmatrix} a & b \\ A & B \end{vmatrix} y_0 = \begin{vmatrix} a & c \\ A & C \end{vmatrix} .$$

The completing the square process shows, among other things, that: if a, b and c are rational numbers and the equation $ax^2 + bx + c = 0$ has a rational root, then $b^2 - 4ac$ is the square of a rational number, and if $a \neq 0$, b, c are rational numbers (real numbers, complex numbers) and $b^2 - 4ac = 0$, then the equation $ax^2 + bx + c = 0$ has exactly one root given by $-b/2a$.

As is indicated by these examples, it is feasible to make precise statements only if the number systems of elementary algebra have been defined. This provides an introduction which prepares the freshman for the idea that college algebra is different from high school algebra (and sends a small rush of students to the deans office at the first of the quarter). When all possible "if - then" statements have been made - and argued - from the very first, the class is in a position to deal with difficulties in some genuinely new material. Since the logical implications of the manipulation of an equation to get x all alone on one side have been pointed out repeatedly, the so-called "extraneous roots" of radical equations appear more as a justification of this care than as a violation of vested rights of computers. (Examples like $x - 7/(x-2) = 2 - 7/(x-2)$ can be given early to show the need for caution). The use of the mathematical induction postulate, which must be stated in an "if - then" form, is in accord with the spirit of the course and may be used to prove such statements as: if a is the first term of an arithmetic progression of common difference d and n is a natural number, then the n-th term is $a + (n-1)d$. In most books, the theory of equations material is already in the form suggested here and the class should have more interest in theorems which depend on the number system at hand. The necessarily informal definition of the real numbers in such a course precludes a mathematical development of logarithms. (I must admit that my classes are taken in by the familiar swindles here).

It is certainly more fun to teach a class of freshmen from the point of view suggested. Some students can construct proofs and still more seem to appreciate arguments. It is reasonable to expect that the students who catch on to what is going on should be able to apply algebra in related courses and that those who really enjoy the work should be encouraged to continue in mathematics.

The University of Georgia

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

Decision Processes, edited by Robert M. Thrall, Clyde H. Coombs, and Robert L. Davis, September, 332 pp. - $5.00 by John Wiley & Sons.

Correlating theory and research in many fields, the new book is the outgrowth of a seminar held at Santa Monica in 1952, under the auspices of the University of Michigan, the RAND Corporation, the Ford Foundation, and the Office of Naval Research. Further research stemming from the original proceedings, as well as additional related papers, have been included in the present study.

Consisting of nineteen papers by twenty-three of the scientists who participated in the seminar, the volume draws on the fields of mathematics, statistics, psychology, economics, and philosophy. The individual contributions range in character from pure mathematics to experiments in group dynamics, but all are directed at the application of mathematics to behavioral sciences in general and at decision processes in particular. After an introduction that discusses the range and interrelationships of the papers, the book goes more deeply into individual and social choice, learning theory, theory and applications of utility, and experimental studies.

The specialists represented in the book are as follows: David Beardslee, Herbert G. Bohnert, R. R. Bush, Clyde H. Coombs, Gerard Debreu, William K. Estes, Leon Festinger, Merrill M. Flood, Leo A. Goodman, Melvin Hausner, Paul Hoffman, G. Kalisch, Douglas Lawrence, Jacob Marschak, John Milnor, Frederick Mosteller, J. Nash, E. D. Nering, Roy Radner, Howard Raiffa, G. L. Thompson, Robert M. Thrall, and Stefan Vail.

Richard Cook

ROUND TABLE ON FERMAT'S LAST THEOREM

THE LAST THEOREM OF FERMAT NOT ONLY A PROBLEM OF ALGEBRAIC ANALYSIS BUT ALSO A PROBABILITY PROBLEM?

Fred G. Elston

I.

In the equation

(1) $$x^r + y^r = z^r ,$$

we regard x^r and y^r as elements of a combination in a series of integers from 1 to n^r. In this series there are n integers of the r^{th} power. The number of combinations is

(2) $$c = \frac{n(n-1)}{2!} .$$

The problem under discussion is: what is the probability that the sum of two elements "hit" and z^r? The series of numbers where such a favorable event could take place is within 1 to $2n^r$, because $n^r + (n-1)^r$ is the highest value that can be reached by addition.

So there are $2n^r$ integers, among them there are $\sqrt[r]{2-n}$ "favorable" integers. The probability in any case to make a "hit" is

(3) $$p = \frac{\sqrt[r]{2\ n}}{2n^r} .$$

The multiplication of c and p will constitute the probable number of cases for (1). Let us call $cp = W$. Then

$$W = \frac{n(n-1)}{2!\ 2^{\frac{r-1}{r}}\ n^{r-1}} .$$

or

(4) $$W = \frac{n-1}{2^{\left(\frac{2r-1}{r}\right)}\ n^{r-2}} .$$

We see that for $r = 2$ and n larger than 3, W is larger than 1. This is consistent with the fact that for $y = 4$ (within the limits of 1 to 4) we have the first case of the Pythagorean numbers

$$3^2 + 4^2 = 5^2 .$$

For r larger than 2, W is smaller than 1, and the value of W approaches 0 as n increases, except for $n = 3$, but here the theorem has been proven.

II.

The improbability of a solution of (1) for r larger than 2 does of course not exclude the possibility of the existence of such a case.

Let us suppose there is just one case of (1) r larger than 3. Then there is an infinite number of cases, for (1) involves

$$v^r x^r + v^r y^r = v^r z^r , \tag{5}$$

v representing all integers from 1 to infinity. That implies that as v becomes larger there will be more and more cases referred to a series from 1 to n_1 -- n also becoming larger and larger.

III.

The statements in I. and II. are contradictory. In I. we came to the conclusion that according to (4) the probability of any cases approaches 0, i.e. the certainty that there are no cases if n becomes infinite.

In II. we saw that, if n is infinite, the number of cases is infinite.

The only conclusion from this contradiction is: that the assumption that there is a case of (1), r larger than 3, is false.

Consequently there is* no solution of the equation

$$x^r + y^r = z^r, \ r \text{ larger than } 3.$$

IV.

If, with increasing n we neglect, in our equation (4), 1 in the numerator, we get

$$W = \frac{1}{2^{\frac{2r-1}{r}} n^{r-3}}$$

or

$$W = \frac{n^{3-r}}{2!\ 2^{\frac{r-1}{r}}} . \tag{6}$$

The numerator shows that, if r is larger than 2, W is smaller than 1. Let us now split z^r in the equation (1), not just in 2, but in any number q of integers of the r^{th} power.

***Should this not read "there is in *all probability* no solution--". What do you think?**

In this Forum on Fermat's Last Theorem, I am presuming to act as moderator. Editor

$$a_1^r + a_2^r + \cdots\cdots a_q^r = z^r \,. \tag{7}$$

Then the combination will be

$$c_q = \frac{n(n-1)(n-2)\cdots\cdots(n-q+1)}{q!} \,. \tag{8}$$

Neglecting again the numbers from 1 to $q - 1$, since n can be considered very large, we have

$$c'_q = \frac{n^q}{q!} \,. \tag{9}$$

The "universe" where a "hit" may be performed is the series from 1 to qn^r. Consequently,

$$W_q = \frac{n^{q+1-r}}{q!\; q^{\frac{r-1}{r}}} \,. \tag{10}$$

Here again the numerator shows that if r is larger than q, W_q is smaller than 1.

We make now the same deductions for the generalized case as in II. and III. and we find the following:

> No integer of the r^{th} power
> can be split into less than
> r integers of the r^{th} power.

Ed's. Comment: Section I appeared in Vol. 28, No. 1, p. 49, essentially as it is here. But it seems best to publish this entire, interesting, paper as a unit.

I am interested in exchanging with fellow mathematicians such items as mathematical fallacies and oddities.

William T. Sandlin
Box 517
Richmond, Kentucky

A REMARK ON FERMAT'S LAST THEOREM

Louis S. Mann

(A). Consider

$$a^n + b^n = c^n$$

where a, b, c and n are positive integers.

(1). Suppose (A) factors into

$$(c^{\frac{n}{2}} - b^{\frac{n}{2}} - a^p)(c^{\frac{n}{2}} + b^{\frac{n}{2}} - a^{n-p}) = 0 .$$

Expanding we have

$$c^n - b^n + a^n - a^{n-p}\,(c^{\frac{n}{2}} - b^{\frac{n}{2}}) - a^p(c^{\frac{n}{2}} + b^{\frac{n}{2}}) = 0.$$

(2). In order that (1) be identical with (A), we must have

$$p = 0,\ c^{\frac{n}{2}} - b^{\frac{n}{2}} = 1,\ c^{\frac{n}{2}} + b^{\frac{n}{2}} = a^n .$$

(3). The simultaneous equations,

$$c^{\frac{n}{2}} - b^{\frac{n}{2}} = 1,\ c^{\frac{n}{2}} + b^{\frac{n}{2}} = a^n$$

yield

$$c^{\frac{n}{2}} = \frac{a^n + 1}{2},\ b^{\frac{n}{2}} = \frac{a^n - 1}{2} .$$

These are solutions of (A) subject to the additional restrictions imposed in (2).

(4). (3) cannot yield solutions for $n > 2$ for

$$c^{\frac{n}{2}} - b^{\frac{n}{2}} = 1,$$

is impossible for $n > 2$ as is seen by putting $c = b + k$, k being positive, and noting that

$$(b + k)^s - b^s > 1 \text{ if } S > 1,$$

however:

(a). For $n = 2$, a odd > 1 (3) yields an infinite number of primitive Pythagorean triplets subject to the restrictions imposed in (2).

Euclid's algorithm will show that

$$\left(a^n, \frac{a^n - 1}{2}\right) = 1.$$

(b). Integers can be found that will satisfy the equality

$$\sum_{i=1}^{n} (a_i)^2 = c^2$$

In order to prove this, I will first prove that any positive odd integer greater than 1 can be represented by the difference of 2 squares.

$$c^2 - a_n^2 = k, \quad k \text{ odd} > 1.$$

In (3) putting

$$a_n = b, \; n = 2, \; k = a^2,$$

we have

$$c^2 = \left(\frac{k+1}{2}\right)^2, \; a_n^2 = \left(\frac{k-1}{2}\right)^2$$

so that

$$c^2 - a_n^2 = k,$$

putting

$$k = a_1^2 + a_2^2 + \cdots + a_{n-1}^2,$$

we have

$$a_1^2 + a_2^2 + \cdots + a_{n-1}^2 + \left(\frac{k-1}{2}\right)^2 = \left(\frac{k+1}{2}\right)^2.$$

(c). Integers can be found that will satisfy the equality

$$\sum_{i=1}^{n} (a_i)^{\frac{1}{r}} = c^{\frac{1}{r}}, \quad r = (1, 2, 3, \cdots m).$$

In (3) putting

$$a_n = b, \; n = \frac{1}{r}, \; k = a^{\frac{1}{r}}, \; k \text{ odd} > 1 \;,$$

we have

$$c^{\frac{1}{r}} = \left(\frac{k+1}{2}\right)^2, \; a_n^{\frac{1}{r}} = \left(\frac{k-1}{2}\right)^2 \;,$$

so that

$$c^{\frac{1}{r}} - a_n^{\frac{1}{r}} = k,$$

putting

$$k = a_1^{\frac{1}{r}} + a_2^{\frac{1}{r}} + \cdots + a_{n-1}^{\frac{1}{r}},$$

we have

$$a_1^{\frac{1}{r}} + a_2^{\frac{1}{r}} + \cdots + \left(\frac{k-1}{2}\right)^2 = \left(\frac{k+1}{2}\right)^2.$$

(d). It has been shown that integer solutions can be found for

$$F + a^2 = c^2, \quad F \text{ odd} > 1,$$

and

$$F + a^{\frac{1}{r}} = c^{\frac{1}{r}}, \; F \text{ odd} > 1, \quad r = (1, 2, 3, \cdots m),$$

where F can be any expression, since a and c are determined by F.

(B). Consider

$$a^n + b^n = c^n,$$

where a, b, c and n are positive integers

(e). Let

$$a^n = u^n v^n .$$

(f). So that

$$u^n v^n + b^n = c^n .$$

(g). Suppose (f) factors into

$$(c^{\frac{n}{2}} - b^{\frac{n}{2}} - v^n)(c^{\frac{n}{2}} + b^{\frac{n}{2}} - u^n) = 0.$$

Expanding we have

$$c^n - b^n + u^n v^n - u^n (c^{\frac{n}{2}} - b^{\frac{n}{2}}) - v^n (c^{\frac{n}{2}} + b^{\frac{n}{2}}) = 0.$$

(h). In order that (g) be identical with (f) we must have

$$c^{\frac{n}{2}} - b^{\frac{n}{2}} = v^n, \quad c^{\frac{n}{2}} + b^{\frac{n}{2}} = u^n .$$

(i). The simultaneous equations in (h) yield

$$c^{\frac{n}{2}} = \frac{u^n+v^n}{2}, \quad b^{\frac{n}{2}} = \frac{u^n-v^n}{2}.$$

(j). $$c^{\frac{n}{2}} = \frac{u^n+v^n}{2}, \quad b^{\frac{n}{2}} = \frac{u^n-v^n}{2}, \quad a^n = u^n v^n.$$

or

$$c^n = (u^n+v^n)^2, \quad b^n = (u^n-v^n)^2, \quad a^n = 4u^n v^n,$$

are solutions of (B).

(k). In particular for $n = 2$ (j) yields the well known solutions

$$c = u^2+v^2, \quad b = u^2-v^2, \quad a = 2uv.$$

(l). The solution

$$a^n = 4u^n v^n,$$

indicates that there is no integer a for $n > 2$.

(m). (3) is, of course, a special case of (j).

Editor's Note:

There should be comments on this paper.

HONORARY MATHEMATICS FRATERNITY

Chapters of Pi Mu Epsilon, national honorary mathematics fraternity, may be chartered only in colleges and universities requiring at least 8 semester hours beyond calculus for a mathematics major and having an average of at least 5 majors per year. The institutions must have at least one Ph. D. in mathematics on its staff and must have had an active mathematics club for at least one year. Faculty members of institutions meeting these requirements may write to Richard V. Andree, Department of Mathematics, The University of Oklahoma, Norman, Oklahoma for further information.

ON THE CASE $n = 3$, OF FERMAT'S LAST THEOREM

Pedro A. Piza

The cube of $a^2 + 3b^2$ is expressed as a number of the same type $p^2 + 3q^2$ by the identity

(1) $$p^2 + 3q^2 = (a^2 + 3b^2)^3 = (a^3 - 9ab^2)^2 + 3(3a^2b - 3b^3)^2 .$$

This identity serves to simplify Euler's classic proof of the impossibility in relatively prime integers z, x, y of $z^3 = x^3 + y^3$, as follows:*

Of the three integers z, x, y, one must be even and the other two odd. We are free to consider x and y odd and to set $x + y = 2p$, $x - y = 2q$, $x = p + q$, $y = p - q$, with p and q coprime, for otherwise x and y and z would have a common factor. Then $z^3 = (p + q)^3 + (p - q)^3 = 2p\,(p^2 + 3q^2)$. Suppose $p \neq 3n$. Then the two factors $2p$ and $p^2 + 3q^2$ are coprime and each is a cube:

$$2p = u^3, \qquad p^2 + 3q^2 = v^3 .$$

If $p^2 + 3q^2$ is a cube it is the cube of an integer of the same type $a^2 + 3b^2$ in accordance with identity (1) as follows:

$$v^3 = p^2 + 3q^2 = (a^2 + 3b^2)^3 = (a^3 - 9ab^2)^2 + 3(3a^2b - 3b^3)^2 ,$$

with a and b coprime, for otherwise p and q would have a common factor. $p = a(a + 3b)(a - 3b)$, $q = 3b(a + b)(a - b)$. Since q is divisible by 3 and must be coprime with p, $p \neq 3n$ as we had supposed. Then in

(2) $$2p = u^3 = 2a\,(a + 3b)(a - 3b) ,$$

a is not divisible by 3 since p is not. Hence the three factors in (2) are coprime and each is a cube:

$$2a = r^3,\; a + 3b = s^3,\; a - 3b = t^3,\; r^3 = s^3 + t^3 ,$$

with

$$(r, s, t) < (z, x, y) .$$

Starting all over again with $r^3 = s^3 + t^3$ we find theoretically another set of integers $(r_1, s_1, t_1) < (r, s, t)$ satisfying

$$r_1{}^3 = s_1{}^3 + t_1{}^3 .$$

*See Dickson's History of the Theory of Numbers, Vol. II, pp. 545-6.

This, repeated, constitutes impossible infinite descent and proves the impossibility of $z^3 = x^3 + y^3$ in integers.

By use of identity (1) the first case of Euler's proof is clarified and the second case in which $p = 3n$ is assumed, is eliminated.

San Juan, Puerto Rico

Ed's. Comment: Many proofs much simpler than Euler's have been published for the case $n = 3$. However, it seems desirable to publish this note to illustrate the method of "Infinite Descent".

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

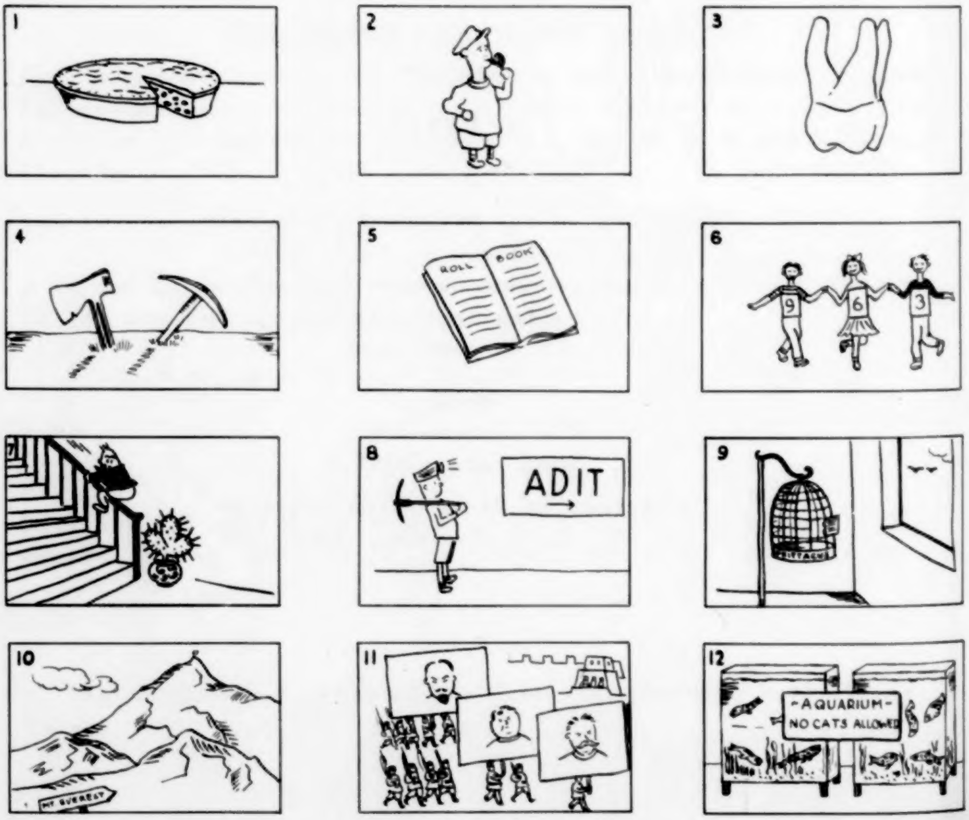
Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 29, California.*

PROPOSALS

222. *Proposed by C. W. Trigg, Los Angeles City College and Leon Bankoff, Los Angeles, California.*

Translate each of the following sketches into a mathematical term.



223. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn.*

Prove that there is no integral triangle such that $\cos A \cos B + \sin A \sin B \sin C = 1$.

224. *Proposed by Ben B. Bowen, Vallejo College, California.*

Find the radius of a circle when a chord, whose maximum distance from the circumference is ten feet, cuts off an arc of 160 feet.

225. *Proposed by P. A. Piza, San Juan, Puerto Rico.*

Find an equality concerning squares of integers in which appear twelve consecutive squares and no others.

226. *Proposed by P. D. Thomas, Eglin Air Force Base, Florida.*

Tangents are drawn from a point P to an ellipse. If R and Q are the points of contact and O is the center of the ellipse, find the locus of P if the area of the quadrilateral $PQOR$ remains constant.

227. *Proposed by Huseyin Demir, Zonguldak, Turkey.*

Let $A_1 B_1$, $A_2 B_2$ and $A_3 B_3$ be three bars of lengths l_1, l_2 and l_3 with weights W_1, W_2 and W_3 respectively. The ends B_1, B_2 and B_3 rest on a horizontal surface while the other ends A_1, A_2 and A_3 are supported by the bars $A_3 B_3$, $A_1 B_1$ and $A_2 B_2$ respectively. Find the reactions R_1, R_2 and R_3 at B_1, B_2 and B_3.

228. *Proposed by Howard D. Grossman, New York, New York.*

Prove that the number of partitions of any number into odd parts greater than unity is equal to the number of partitions into two or more equal parts of which the two largest differ by unity.

ERRATA

In the problem **206** [May 1954] line three should read... $2\gamma = \alpha + \beta +$ constant ...

SOLUTIONS

A Radical Cryptarithm

201. [May 1954] *Proposed by Leon Bankoff, Los Angeles, California.*

Solve the following cryptarithm. The first four convergents in the continued fraction expansion of $\sqrt{**}$ are

$$\frac{*}{*}, \frac{**}{**}, \frac{***}{***}, \frac{****}{****}.$$

where the asterisks are integers.

Solution by S. H. Sesskin, Hofstra College, New York. Expansion of $\sqrt{p^2 + a}$ yields the following convergents:

$$\frac{p}{1}, \frac{2p^2 + a}{2p}, \frac{4p^3 + 3ap}{4p^2 + a}, \frac{8p^4 + 8ap^2 + a^2}{8p^3 + 4ap}.$$

It is evident from $\sqrt{**}$ that p must be an integer from 3 to 9. Since $(2p^2 + a)/2p = **/**$, then we must have $p \geq 5$ to yield two digits in the denominator of the convergent. In the numerator $2p^2 + a = **$ or $2p^2 = ** - a$ equals at most 98 restricting p further to 7, 6 and 5.

The second convergent is derived from $p + a/2p$, or

$$p + \cfrac{1}{\cfrac{2p}{a}} \cdots$$

and since the fractions are continued by taking the greatest integer in $[2p/a]$, then the elementary form of the second convergent is

$$5 \text{ or } 6 \text{ or } 7 + \cfrac{1}{\cfrac{10 \text{ or } 12 \text{ or } 14}{a}}$$

In this form it is evident that a must be 1 in order to provide a two-digit denominator.

Using the above we test $\sqrt{25 + 1}$, $\sqrt{36 + 1}$ and $\sqrt{49 + 1}$. The $\sqrt{26}$ yields the convergents 5/1, 51/10, 515/101, 5201/1020.

The solution is unique as $\sqrt{37}$ yields a fourth convergent *****/**** while $\sqrt{50}$ yields a third convergent ****/***.

Also solved by Richard K. Guy, University of Malaya; M. S. Klamkin, Polytechnic Institute of Brooklyn; E. P. Starke, Rutgers University, and the proposer.

Center of Curvature

202. [May 1954] *Proposed by Chih-yi Wang, University of Minnesota.*

Find the coordinates of the center of curvature of:

$$y = x^x \sin x(\text{arc cot } x) \log x \quad \text{at the point } (1,1).$$

I. *Solution by George Mott, Mineola, New York.* In the expression $y = x^{x \sin x (\text{arc} \cot x) \log x}$, $y(1) = 1$. Differentiating we have: $y' = x^{x \sin x(\text{arc} \cot x) \log x}\,[2 \sin x\,(\text{arc} \cot x) \log x + \sin x\,(\text{arc} \cot x) \log^2 x + x \cos x\,(\text{arc} \cot x) \log^2 x - (x \sin x \log^2 x)/(1 + x^2)]$. Here $y'(1) = 0$.

Let $g(x) = x^{x \sin x(\text{arc} \cot x) \log x}$ and $f(x)$ represent the ·ression in brackets. Differentiating $y'(x) = g(x)\, f(x)$ we have $y''(x) = g'(x)\, f(x) + g(x)\, f'(x)$. Now $g'(1) = 0$, $f(1) = 0$, $g(1) = 1$ and $f'(1) = \frac{\pi}{2} \sin 1$ so $y''(1) = g'(1)\, f(1) + g(1)\, f'(1) = \pi/2 \sin 1$.

The coordinates (h,k) of the center of curvature are given by:

$$h = x_1 - \frac{y'(x_1)\,\left[1 + y'(x_1)^2\right]}{y''(x_1)}$$

$$k = y_1 + \frac{1 + y'(x_1)^2}{y''(x_1)}$$

where

$$(x_1,\ y_1) = (1,\ 1).$$

Hence

$$h = 1 - 0 = 1 \text{ and } k = 1 + \frac{1}{\frac{\pi \sin 1}{2}}\ .$$

II. *Solution by the proposer.* Taking the logarithm of both sides we get $\log y = x \sin x \text{ Arc} \cot x\,(\log x)^2$. Differentiating with respect to x twice we obtain respectively:

$$y^{-1} \frac{dy}{dx} = F(x)\,(\log x)^2 + 2 \sin x \text{ Arc} \cot x \log x\ ,$$

and

$$-y^{-2} \left(\frac{dy}{dx}\right)^2 + y^{-1} \frac{d^2y}{dx^2} = F'(x)\,(\log x)^2 + \frac{2\,F(x)}{x} \log x +$$

$$G(x) \log x + \frac{2 \sin x \text{ Arc} \cot x}{x}$$

where $F(x)$, $F'(x)$ and $G(x)$ are continuous at $x = 1$. Hence

$$\left.\frac{dy}{dx}\right|_{(1,1)} = 0 \text{ and } \left.\frac{d^2y}{dx^2}\right|_{(1,1)} = \frac{\pi}{2} \sin 1.$$

The radius of curvature follows immediately as

$$\frac{2}{\pi \sin 1} .$$

Since the curve has a horizontal tangent at (1,1) and is concave upward there, the required center is

$$(1,\ 1 + \frac{2}{\pi \sin 1}) .$$

Also solved by S. H. Sesskin, Hofstra College, New York.

Circle Trisection

203. [May 1954] *Proposed by Norman Anning, Alhambra, California.*

Prove that three of the intersections of $x^2 - y^2 + ax + by = 0$ and $x^2 + y^2 - a^2 - b^2 = 0$ trisect the circle through these three points.

I. *Solution by W. O. Moser, University of Toronto.* In polar coordinates $(\rho,\ \theta)$ these curves have the equations $\rho(\rho \cos 2\theta + a \cos \theta + b \sin \theta) = 0$ and $\rho^2 = a^2 + b^2$. The values of θ at the points of intersection are the solutions of the equation

$$a \cos \theta + b \sin \theta = \sqrt{a^2 + b^2}\ \cos 2\theta .$$

Define ψ so that

$$\cos \psi = \frac{a}{\sqrt{a^2 + b^2}} \quad \text{and } \sin \psi = \frac{b}{\sqrt{a^2 + b^2}}$$

Then

$$\cos \psi \cos \theta + \sin \psi \sin \theta = \cos 2\theta .$$

I. e.

$$\cos (\psi - \theta) = \cos 2\theta .$$

The four essentially different solutions of this equation are

$$\frac{\psi}{3},\ \frac{\psi}{3} + \frac{2\pi}{3},\ \frac{\psi}{3} + \frac{4\pi}{3},\ -\psi .$$

The first three of these satisfy the conditions of the problem.

II. *Solution by Huseyin Demir, Zonguldak, Turkey.* Set $r^2 = a^2 + b^2$ and let the value of y obtained by adding together the two equations be substituted in the first equation. We get an equation:

$$4x^4 + 4a x^3 - 3r^2x^2 - 2ar^2x + a^2r^2 = 0,$$

of fourth degree in x of which the roots are x_1, x_2, x_3, x_4.

If the triangle $A_1A_2A_3$ corresponding to x_1, x_2, x_3 is equilateral, $x_1 + x_2 + x_3$ will vanish (for $A_1A_2A_3$ is in the circle $x^2 + y^2 - r^2 = 0$ centered at 0), and x_4 is from the second coefficient $(x_1 + x_2 + x_3) + x_4 = x_4 = -a$.

Therefore to prove the statement it will suffice to show that the above equation is divisible by $x + a$ and that in the quotient obtained the term x^2 is missing.

By division we get

$$4x^4 + 4ax^3 - 3r^2x^2 - 2ar^2x + a^2r^2 = (x + a)(4x^3 - 3r^2x + ar^2),$$

and this is in agreement with what we said above. Hence $A_1A_2A_3$ is an equilateral triangle.

III. *Solution by Richard K. Guy, University of Malaya, Singapore.* The curves are a rectangular hyperbola and a circle, centre 0. The circle through 3 of the points of intersection is therefore the circle $x^2 + y^2 = a^2 + b^2$ by inspection, $(-a, b)$ is common to the 2 curves. Let P, Q, R be the other 3 points of intersection. Then, by well-known theorems, the orthocentre of PQR and the fourth point, $(-a, b)$, of intersection of the rectangular hyperbola with the circle PQR, lie at opposite ends of a diameter at the rectangular hyperbola. But the centre of the rectangular hyperbola is $(-\tfrac{1}{2}a, \tfrac{1}{2}b)$. Therefore the orthocentre of PQR is 0. But this is also the circumcentre. Therefore PQR is equilateral.

IV *Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn.* Consider

$$z^2 = (a^2 + b^2)^{3/2} \; e^{i\theta} \quad \text{where } \cos\theta = \frac{-a}{\sqrt{a^2 + b^2}},$$

and

$$\sin\theta = \frac{-b}{\sqrt{a^2 + b^2}}.$$

Then

$$x^2 + y^2 = a^2 + b^2,$$

and

$$z^2 = (a^2 + b^2)^{3/2} \frac{(\cos\theta + i\sin\theta)}{z}.$$

Equating the real parts of this equation leads to $x^2 - y^2 = -ax - ay$. Thus the solution follows immediately.

Also solved by Barney Bissinger, Lebanon Valley College, Pennsylvania; Rex D. Depew, Vanderbilt University; M. S. Klamkin, Polytechnic Institute of Brooklyn (a second solution); S. H. Sesskin, Hofstra College, New York; N. Shklov, University of Saskatchewan; C. W. Trigg, Los Angeles City College; Chih - yi Wang, University of Minnesota; Hazel S. Wilson, Jacksonville State College, Alabama and the proposer.

Cevian Triples

204. [May 1954] *Proposed by C. W. Trigg, Los Angeles City College.*

In the triangle ABC let the feet of the median (m_a), of the internal angle bisector (t_a), of the cevian (p_a) to the contact point of the incircle with a, and of the cevian (q_a) to the contact point of the excircle relative to A with a be respectively A_m, A_t, A_p and A_q. Use similar notation for the corresponding lines to b and c.

1). Determine the relationship between the sides of the triangle if the following triads are to be concurrent: p_a, m_b, t_c at S; p_z, q_b, m_c at R; m_a, p_b, t_c at T; q_a, p_b, m_c at V.

2). Show that A_pB_q and A_qB_p are parallel to AB; C_tB_p and SV are parallel to BC; and C_tA_p and RT are parallel to AC.

Solution by Huseyin Demir, Zonguldak, Turkey. 1). We determine the positions of the cevians u_a, v_b, w_c or their feet A_u, B_v, C_w on the respective sides BC, CA, AB by the ratios:

$$k(A_u) = A_uB/A_uC, \quad k(B_v) = B_vC/B_vA, \quad k(C_w) = C_wA/C_wB$$

Since these points are interior points of the sides all these ratios are negative. Their values are tabulated below:

$$k(A_m) = -1,\ k(A_t) = -c/b,\ k(A_p) = -(s-b)/(s-c),\ k(A_q) = -(s-c)/(s-b)$$

$$k(B_m) = -1,\ k(B_t) = -a/c,\ k(B_p) = -(s-c)/(s-a),\ k(B_q) = -(s-a)/(s-c)$$

$$k(C_m) = -1,\ k(C_t) = -b/a,\ k(C_p) = -(s-a)/(s-b),\ k(C_q) = -(s-b)/(s-a)$$

Now, the required common condition is obtained by applying Ceva's theorem to the triples of cevians:

TRIPLES:	POINTS:	CEVA THEOREM:	CONDITIONS:
$p_a,\ m_b,\ t_c$	S	$[-(s-b)/(s-c)][-1][-b/a]=-1$	$(s-b)/(s-c)=a/b$
$p_a,\ q_b,\ m_c$	R	$[-(s-b)/(s-c)][-(s-a)/(s-c)][-1]=-1$	$(s-a)(s-b)=(s-c)^2$
$m_a,\ p_b,\ t_c$	T	$[-1][-(s-c)/(s-a)][-b/a]=-1$	$(s-c)/(s-a)=a/b$
$q_a,\ p_b,\ m_c$	V	$[-(s-c)/(s-b)][-(s-c)/(s-a)][-1]=-1$	$(s-c)^2=(s-a)(s-b)$

These four conditions just obtained are easily seen to be identical with the unique condition

$$c = (a^2 + b^2)/(a + b) \ .$$

2) (a): To prove $A_pB_q//A_qB_p//AB$ we see that $k(A_p) = 1/k(B_q)$, $k(A_q) = 1/k(b_p)$.

(b): To prove $C_tB_p//BC$ we similarly see $k(B_p) = 1/k(C_t)$ (see cond (3)).

Now to prove $SV//BC$ we apply the Menelaus theorem to the triangles BCB_m, BCC_m cut respectively by the lines ASA_p, AVA_q:

$$(A_pB/A_pC)(AC/AB_m)(SB_m/SB) = 1, \quad \text{then} \quad SB/SB_m = 2k(A_p) \ .$$

$$(A_qB/A_qC)(VC/VC_m)(AC_m/AB) = 1, \quad \text{then} \quad VC/VC_m = 2/k(A_q).$$

Hence

$$SB/SB_m = 2k(A_p) = 2/k(A_q) = VC/VC_m \ .$$

This proves that S, V divide BB_m, CC_m in the same ratio. But having $B_mC_m//BC$ the property follows.

(c): To prove $C_tA_p//AC$ we see that $k(C_t) = 1/k(A_p)$.

Then finally to show $RT//AC$ we again apply the Menelaus theorem to the triangles CAC_m, CAA_m cut by the lines BRB_q, BTB_q respectively.

$$k(B_q)\ (BA/BC_m)(RC_m/RC) = 1 \quad \text{then} \quad RC/RC_m = 2k(B_q) \ ,$$

$$k(B_p)\ (TA/TA_m)(BA_m/BC) = 1 \quad \text{then} \quad TA/TA_m = 2/k(B_p),$$

and

$$RC/RC_m = 2k(B_q) = 2/k(B_p) = TA/TA_m \ .$$

Hence R and T divide CC_m, AA_m in the same ratio. But having $C_mA_m//CA$ we also have $RT//CA$.

Q. F. D.

Also solved by Sister M. Stephanie, Georgian Court College, N. J. and the proposer.

A Circle Concentric With A Polygon

205. [May 1954] *Proposed by Victor Thebault, Tennie, Sarthe, France.*

The locus of the point, the sum of the products of whose distances from the pairs of opposite sides of a regular polygon of $2n$ sides is constant is a circle concentric with the polygon.

Solution by R. D. Depew and H. R. Rouse, Vanderbilt University, Nashville, Tennessee. Let A_j be the vertices and $a_j = A_j A_{j+1}$ be the sides of the regular polygon, where $j = 1, \ldots, 2n$ and $A_{2n+1} = A_1$. Let d_j be the directed distance from a variable point P to a_j.

The given restriction on P may be stated as

$$(1) \qquad d_j d_{j+n} = k, \ k \text{ const.}$$

All summations will be understood to range from 1 to n.

Take the center 0 of the polygon as the origin of Cartesian coordinates and let the positive x-axis be the perpendicular from 0 to a_1.

Using the normal form of the equation of the straight line, we obtain

$$(2) \qquad d_j = x \cos\frac{(j-1)\pi}{n} + y \sin\frac{(j-1)\pi}{n} - p$$

$$d_{j+n} = -x \cos\frac{(j-1)\pi}{n} - y \sin\frac{(j-1)\pi}{n} - p .$$

Taking the coordinates of P as (x,y) we obtain from (2)

$$\Sigma\, d_j d_{j+n} = \Sigma\left[p^2 - \left(x \cos\frac{(j-1)\pi}{n} + y \sin\frac{(j-1)\pi}{n}\right)^2\right]$$

and from (1)

$$(3)\ n\, p^2 - x^2\, \Sigma \cos^2\frac{(j-1)\pi}{n} - xy\, \Sigma\, 2 \sin\frac{(j-1)\pi}{n} \cos\frac{(j-1)\pi}{n}$$

$$- y^2\, \Sigma \sin^2\frac{(j-1)\pi}{n} = k .$$

We note that

$$(4) \qquad 2 \sin \frac{(j - 1)\pi}{n} \cos \frac{(j - 1)\pi}{n} = \sin \frac{2(j - 1)\pi}{n} .$$

Recalling that

$$(5) \qquad \exp(i\theta) = \cos\theta + i \sin\theta$$

and noting that

$$\Sigma \exp \frac{(2(j - 1)\pi i)}{n}$$

is a finite geometric series with ratio

$$\exp \frac{(2\pi i)}{n} ,$$

we obtain

$$(6) \quad \Sigma \cos \frac{2(j - 1)\pi}{n} + i \Sigma \sin \frac{2(j - 1)\pi}{n} = \exp\left(\frac{2(j - 1)\pi}{n}\right)$$

$$= \frac{1 - e^{2\pi i}}{1 - e^{2\pi i/n}} = 0 .$$

whence

$$(7) \qquad \Sigma \cos \frac{2(j - 1)\pi}{n} = \Sigma \sin \frac{2(j - 1)}{n} = 0.$$

Further, from (7) we obtain

$$(8) \qquad \begin{aligned} \Sigma \sin^2 \frac{(j - 1)\pi}{2} &= 1/2\, \Sigma \left(1 - \cos \frac{2(j - 1)\pi}{n}\right) = \frac{n}{2} . \\ \Sigma \cos^2 \frac{(j - 1)\pi}{2} &= 1/2\, \Sigma \left(1 + \sin \frac{2(j - 1)\pi}{n}\right) = \frac{n}{2} ; \end{aligned}$$

It follows from (3), (4), (7), and (8) that the locus Γ of P has the equation

$$(9) \qquad x^2 + y^2 = 2\,(p^2 - k/n),$$

which is a circle concentric with the polygon.

It is of interest to note the following:

When $k = n\, p^2$, Γ is a single point, the center of the polygon.

When $$\frac{np^2}{2} < k < n\,p^2\,,$$

Γ is interior to the polygon.

When $$k = \frac{n\,p^2}{2}\,,$$

Γ is the inscribed circle of the polygon.

When $$\frac{n\,p^2}{2}\left(1 - \tan^2\frac{\pi}{2\,n}\right) < k < \frac{n\,p^2}{2}\,,$$

Γ cuts the polygon.

When $$k = \frac{np^2}{2}\left(1 - \tan^2\frac{\pi}{2n}\right),$$

Γ is the circumscribed circle of the polygon.

When $$k > \frac{n\,p^2}{2}\left(1 - \tan^2\frac{\pi}{2n}\right),$$

Γ is exterior to the polygon.

As $k \to -\infty$ the radius of $\Gamma \to \infty$ and when $k > n\,p^2$, Γ is imaginary.

Also solved by Howard Eves, University of Maine; Huseyin Demir, Zonguldak, Turkey; F. C. Gentry, University of New Mexico; E. V. Greer, Bethany - Peniel College, Oklahoma; Richard K. Guy, University of Malaya, Singapore; M. S. Klamkin, Polytechnic Institute of Brooklyn; Joseph D. E. Konhauser, Pennsylvania State University; D. L. Mac Kay, New York, New York; W. O. Moser, University of Toronto; T. F. Mulcrone, St. Charles College, Louisiana; S. H. Sesskin, Hofstra College, New York; Chih - yi Wang, University of Minnesota and the proposer.

Integer Triples

207. [May 1954] *Proposed by P. A. Piza, San Juan, Puerto Rico.*

If $a^2 + b^2 = c^2$ in a Pythagorean triangle whose sides are integers, solve for positive integers x, y and z the equation:

$$(a+b+x)^2 + (a+b+y)^2 = (a+b-z)^2 + (3a+3b+3c)^2$$

Solution by E. P. Starke, Rutgers University. Choose arbitrarily four positive integers k, l, m, n such that $km + ln = 3a + 3b + 3c$ and (in order that $y > 0$) with $km < a + b + 3c/2$.

Then
$$x = kl + mn - a - b,$$
$$y = ln - km - a - b, \qquad z = a + b \pm (kl - mn),$$

satisfy the given equation identically (whether or not a, b, c form a Pythagorean triple.)

Also solved by M. S. Klamkin, Polytechnic Institute of Brooklyn; Sam Kravitz, East Cleveland, Ohio and the proposer.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and source, if known.

Q 128. Determine the probability that a random rational fraction a/b is irreducible. [*Submitted by M. S. Klamkin.*]

Q 129. Each face of a regular dodekahedron is painted with a different color. Using the same twelve colors, how many dodekahedrons with different color arrangements are possible? [*Submitted by Clarence R. Perisho.*]

Q 130. Solve

$$x + y + z = 2$$
$$x^2 + y^2 + z^2 = 14$$
$$x^4 + y^4 + z^4 = 98$$

[*Submitted by Paul H. Yearout.*]

Q 131. Evaluate

$$\lim_{x \to 0} \frac{\tan n x}{x}$$

[*Submitted by Charles Salkind.*]

Q 132. If A, B, C are the angles of a triangle show that $\sin^2 A + \sin B \sin C \cos A$ is symmetric in A, B and C. [*Submitted by M. S. Klamkin.*]

Q 133. If
$$f_1(x) = \frac{1}{\left[\frac{1}{x}\right]}, \quad 0 < x \leq 1,$$
find
$$F(x) = \lim f_n(x)$$
where
$$f_n(x) = f_1(f_{n-1}(x)).$$

[*Submitted by Barney Bissinger.*]

ANSWERS

A 130. Take

$$2(x^4 + y^4 + z^4) - (x^2 + y^2 + z^2)^2 = 2(98) - 14^2 = 0,$$

or

$$x^4 + y^4 + z^4 - 2x^2y^2 - 2x^2z^2 - 2y^2z^2 = 0,$$

which factors into

$$(x + y + z)(x + y - z)(x - y + z)(- x + y + z) = 0.$$

Now

$$x + y + z = 2,$$

$$x + y - z = 0 \quad \text{leads to } z = 1.$$

Hence

$$x + y = 1,$$

and

$$x^2 + y^2 = 13 .$$

These lead to $x = 3$, $y = -2$ or $x = -2$, $y = 3$. Using other factors will lead to permutations of these solutions.

A 131. We note that for small angles $\tan n\, x \approx n\, x$. Hence

$$\lim_{x \to 0} \frac{\tan n\, x}{x} \approx \lim_{x \to 0} \frac{nx}{x} = n .$$

A 132. Since

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{bc \sin A}{abc} = \frac{2\Delta}{abc},$$

where Δ is the area of the triangle and as

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

the given sum equals

$$\frac{2\, \Delta^2(a^2 + b^2 + c^2)}{(abc)^2} .$$

A 133. If m is a positive integer, then for

$$\frac{1}{m + 1} < x \leq \frac{1}{m} ,$$

we have

$$f_n(x) = f_2(x) = f_1(x) = \frac{1}{m}$$

So

$$F(x) = f_1(x)$$

A 128. Probability

$$p = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) \cdots = \Pi\left(1 - \frac{1}{p_r^2}\right)$$

where the infinite product is extended over all primes.

$$\frac{1}{p} = \frac{1}{\Pi\left(1 - \frac{1}{p_r^2}\right)} = \left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \cdots\right)\left(1 + \frac{1}{3^2} + \frac{1}{3^4} + \cdots +\right) \cdots =$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \pi^2/6 .$$

Thus $p = 6/\pi^2$.

A 129. The total number would be the permutations of 12 things divided by the symmetries of the dodekahedron which number 60. Thus we have

$$\frac{12!}{60} = 7983360 .$$

TRICKIES

A trickie is a problem whose solution depends upon the perception of the key word, phrase or idea rather than upon a mathematical routine. Send us your favorite trickies.

T 14. A flexible cable of length twelve feet is hanging from two points at the same height. If the dip in the cable is six feet determine the span. [*Submitted by M. S. Klamkin.*]

T 15. The number 1,001,050,511 in a certain system of enumeration means 147 in the base ten. Give an explanation of the system. [*Submitted by V. C. Harris.*]

SOLUTIONS

S 14. It follows immediately that the span must be zero.

S 15. The number is written in the Roman system of enumeration with arabic numerals. It would normally be written CXLVII.

SEMI-POPULAR AND POPULAR PAGES

HOW MODERN MATHEMATICAL CONCEPTS SHED LIGHT ON ELEMENTARY MATHEMATICS*

Richard V. Andree

My four year old son says that the best stories begin "once upon a time". Thus, I begin,--once upon a time, I was a high school student. My high school teachers never showed me how to multiply 2 by 2 matrices. To this day, I haven't forgiven them for that omission! I know *why* they didn't show me how to multiply 2 by 2 matrices, and for that reason I'm going to show you how it is done.

To begin with, let me tell you what a matrix is. A matrix is a square array of numbers with certain rules for addition and multiplication.

Addition is elementwise:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

But multiplication follows a special "row by column" rule

$$\underset{(A)}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}} \times \underset{(B)}{\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}} = \begin{pmatrix} 1 \times 5 + 2 \times 7 & 1 \times 6 + 2 \times 8 \\ 3 \times 5 + 4 \times 7 & 3 \times 6 + 4 \times 8 \end{pmatrix} = \underset{(AB)}{\begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}}$$

Thus the element in the first (horizontal) row and second (vertical) column of the product *AB* is a sum of elements each of which is the product of an element from the first row of *A* multiplied by the corresponding element from the second column of *B*.

$$\begin{pmatrix} \to 1 & \to 2 & \to \\ \cdot & \cdot & \end{pmatrix} \begin{pmatrix} & \downarrow \\ \cdot & 6 \\ & \downarrow \\ \cdot & 8 \\ & \downarrow \end{pmatrix} = \begin{pmatrix} \cdot & 1 \times 6 + 2 \times 8 \\ \cdot & \cdot \end{pmatrix}$$

*A talk delivered at the 1954 Summer Conference for Mathematics Teachers at the University of Oklahoma.

If we form the product in reverse order we obtain

$$BA = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 5\cdot1 + 6\cdot3 & 5\cdot2 + 6\cdot4 \\ 7\cdot1 + 8\cdot3 & 7\cdot2 + 8\cdot4 \end{pmatrix} = \begin{pmatrix} 23 & 34 \\ 31 & 46 \end{pmatrix}.$$

Thus, in this system, A times B is not the same as B times A! If you tell students about a mathematical system in which $AB \neq BA$, the question, "What good is it?" should soon arise, and usually does. Before we find answers to this question, let us perform an experiment.

Place two closed books on the table in front of you with their faces upward and their spines (bound edge) on the left. This is the normal position in which a book might lie before it was opened. The books will remain closed throughout the experiment.

Rotate the first book through 90° (a right angle) about its bottom edge. (It will now be standing upright on the table.) Now rotate the same book through 90° about its spine. Leave the book in this position.

Rotate the second book through 90° about its spine. (If the book were released at this point it would fall open in reading position.) Now rotate it through 90° about its bottom edge.

Note that the two books are not in the same final position. Each book has been rotated through 90° about its bottom edge and 90° about its spine, but the order was not the same and the results are different. In the physical world in which we live $AB \neq BA$. It is possible to use matrix theory to forecast the result of these and much more complicated problems involving rotations in three dimensional, four dimensional, or higher dimensional space.

If you look in our mathematics library for books on matrix theory, you will find one of the better books is *Elementary Matrices* by Frazier, Duncan and Collar. One of the most interesting sidelights is that none of the three authors is a professional mathematician! They are all *aeronautical engineers*. They wrote the book because aeronautical engineers of today use matrices as an everyday tool of their trade. Flutter analysis, inter-stress computations and complex weighting factors are all more readily handled through the use of matric methods.

If you listen to the conversation going on as you pass the physics department you may well hear discussions involving the Pauli Matrices which are used in electron spin theory and quantum mechanics. Differential equations lie at the heart of much of

today's applied mathematics, and the solution of differential equations is greatly aided by the "eigen werte" of matric theory.

If all this sounds a bit esoteric to you, let's come down to a simpler application. In the design and testing procedures of modern rockets, systems of 36 equations in 36 unknowns frequently occur. These are solved, using machines, by matric methods. I mean *matrices, not determinants*- please don't confuse the two .

You will find matrices form the basis of an important branch of psychology known as factor analysis. Statistics, electrical network theory, oscillation and vibration theory, and circuit analysis are all simplified by the use of matrices--and this is only the beginning!

I hope that you will show your students a little about this fascinating and useful branch of modern mathematics.

PART II.

Now, I'm going to talk about a system having *only seven numbers* 0, 1, 2, 3, 4, 5, 6. What can you do with only seven numbers? You can do almost anything you can do in ordinary arithmetic. The secret is that if we ever arrive at a number other than 0, 1, 2, 3, 4, 5, 6 we simply subtract (or add) a multiple of 7 which will give us a number in our system. Saying it in another way, two numbers are equal mod 7 if they yield the same remainder when divided by seven. $a \equiv b \bmod 7$ if $a = b + k\cdot 7$ for some whole number k. To remind you that we are using special rules I'll write $\equiv$ for equality in place of the usual =.

addition $3 + 4 \equiv 0 \mod 7$*

subtraction $3 - 4 \equiv 6 \mod 7$

You can see that negative numbers are unnecessary.

multiplication $3 \times 4 \equiv 5 \mod 7$

division $3 \div 4 \equiv ?$

Let us examine exactly what we mean by the fraction $3 \div 4$. It is the unique solution of the equation $4x = 3$. Similarly we define the mod 7 solution of $4x \equiv 3$ to be $3 \div 4$ in our new system.

$$\left.\begin{aligned} 4x &\equiv 3 \\ x &\equiv 6 \end{aligned}\right\} \bmod 7 \qquad \text{(Try it and see, } 4\cdot 6 = 24 \equiv 3, \bmod 7.)$$

Thus $3 \div 4 \equiv 6$.

*See "Congruence" by E. L. Dimmick, M.D., Vol. 28, No. 1, Sept.-Oct., 1954.

Isn't this an interesting system? It permits us to do division without using fractions--indeed the system contains no fractions!

Let us examine the quadratic equation $x^2 + x + 5 \equiv 0$. Since there are only seven numbers in this system, we may solve the equation by direct substitution, finding that $x \equiv 1, 5$ are solutions. (Note that they are *not* imaginary numbers.) On the other hand the quadratic equation $x^2 + 2x + 5 \equiv 0$ has no solutions whatsoever in the mod 7 system.

By studying this very simple system you may gain great insight into the heart of arithmetic and algebra. Even more can be gained by also studying the mod 6 system. Mod 6 system contains only six numbers 0, 1, 2, 3, 4, 5, and $a \equiv b$ mod 6 if $a = b + k6$ for some whole number k. The mod 7 system obeys the usual laws of algebra but the mod 6 system violates the cardinal rule by which algebraic equations are solved, namely $AB = 0$ if *and only if* $A = 0$ or $B = 0$. Note that $4 \not\equiv 0$, and $3 \not\equiv 0$, but $4 \times 3 \equiv 12 \equiv 0$ mod 6. The quadratic equation $2x^2 + 4 \equiv 0$ has solutions $x \equiv 1, 2, 4, 5$ (bargain day) while $x^2 + 2 \equiv 0$ has only $x \equiv 2, 4$ as solutions. I trust you will carry out further experiments in these interesting modern number systems. Your students may enjoy them while discovering some of the fundamental concepts behind their algebraic and arithmetic manipulation.

The University of Oklahoma

THE EGG PROBLEM

B. H. Bissinger[1]

One night recently the junior author returned home to find his wife listening to the hackneyed corn of a salesman intent on disposing of a national brand electric sweeper. With the doom of failure becoming apparent, the salesman proposed the following problem and promised a free cleaner for its solution if he would be promised the purchase of one in case it stumped his would-be listeners:

> Three boys had 10, 30, and 50 eggs respectively. They each sold their eggs at the same rate and received the same amount of money. How much did they sell their eggs for?

To such a question a mathematician answers that there is no solution other than the trivial zero solution. The salesman immediately replied with "the" solution of 5 cents for each lot of 7 eggs and 15 cents for each egg left over, totalling 50 cents for each boy.

Of course, it would have been more precise for the salesman to say each boy sold some of his eggs at one rate and the rest at another rate. Is the statement now free of ambiguity? In the May 26, 1954, issue of the New York Times, it is told how the educator, Louis S. Mills, in 1897, flunked the arithmetic test when he applied for the teaching job of the little one-room red school house in Woodstock, Connecticut. It was the problem stated above that he tripped over. In the New York Times article, the editor pictures Professor Fehr of Columbia University with his solution of the problem. However, it is not clear from the editorial that all interpretations of the statement of the problem are considered.

Accepting the fact that the problem is badly stated, we consider the situation that admits *two* different rates, each boy selling *some* of his eggs at X cents apiece and *the rest* of them at Y cents apiece, $0 < Y < X$. The words "some" and "the rest" are interpreted to include the possibilities of "all" or "none". The values X and Y are integers. We wish to show that a solution in the sense of the salesman, i.e., a solution in which one rate can be replaced by a lot rate, always corresponds to a solution in the sense Fehr proposed, but not vice versa. For a given set of numbers of eggs for each boy (10, 30, and 50 in the original problem) there will

[1]The author wishes to thank Professor W. B. Carver of Cornell University for rendering valuable suggestions.

be usually many more solutions a la Fehr than a la salesman. All letters will represent integers and hence all equations are Diophantine. Only elementary algebra will be used to exploit rigorously what a student might call hard in a simple problem.

Generally, let A, B, C be the number of eggs which the three boys have to sell, respectively. Each boy sells *some* eggs D, E, F at Y cents apiece and *the rest* $A-D$, $B-E$, $C-F$, respectively, at X cents apiece. Our problem is to find D, E, F, X, and Y for a given set A, B, C. No loss of generality comes from assuming

$$(1) \qquad 0 < A < B < C$$

$$(2) \qquad 0 < Y < X .$$

Further, the problem requires that D, E, F, X, Y satisfy the conditions

$$(3) \qquad DY + (A-D)X = EY + (B-E)X = FY + (C-F)X$$

$$(4) \qquad 0 \leqq D \leqq A, \quad 0 \leqq E \leqq B, \quad 0 \leqq F \leqq C .$$

The homogeneous appearance of X and Y in (3) turns our interest to their ratio for which we obtain

$$\frac{Y}{X} = \frac{(E-D)-(B-A)}{E-D} = \frac{(F-E)-(C-B)}{F-E} = \frac{(F-D)-(C-A)}{F-D}$$

or

$$(5) \qquad \frac{Y}{X} = 1 - \frac{B-A}{E-D} = 1 - \frac{C-B}{F-E} = 1 - \frac{C-A}{F-D} .$$

Then, because of (1) and (2), we have

$$(6) \qquad 0 < B-A < E-D, \quad 0 < C-B < F-E, \quad 0 < C-A < F-D$$

so that

$$(7) \qquad 0 \leqq D < E < F$$

$$(8) \qquad 0 \leqq C-F < B-E < A-D .$$

As the number of eggs increases from boy to boy, horse sense, as well as (7) and (8), tells us that fewer eggs must be sold at the higher price X and more eggs must be sold at the lower price Y. We can now revise (4) to read

$$(4') \qquad 0 \leqq D < A, \quad 0 < E < B, \quad 0 < F \leqq C .$$

So the boy with the least number of eggs may have to sell all of them at the higher price, but may not sell all of them at the lower price. Similarly, the boy with the most eggs may not sell all

of them at the higher price but may have to sell all of them at the lower price.. Further interpretation of (4′) tells us that the boy "in between" with B eggs may not sell all his eggs at either price. From (5) we have

$$(B-A)(F-D) = (C-A)(E-D) .$$

If G is the highest common factor of $B-A$ and $C-A$, then

$$B-A = HG, \quad C-A = KG, \quad G \geqq 1, \quad 0 < H < K, \quad (H|K) = 1 , \tag{9}$$

and substitution in (5) yields $H(F-D) = K(E-D)$. Since H is prime to K, we have

$$E-D = HT, \quad F-D = KT \tag{10}$$

and because of (6) and (9) $1 \leqq G < T$. From (10) we see a possible maximum of KT would arise from the maximum of F and the minimum of D, namely C. Therefore $KT < C$, and we can amend the last inequality to read

$$1 \leqq G < T \leqq \frac{C}{K} . \tag{11}$$

We note that unless $\frac{C}{K} \geqq 2$, there will be no T satisfying (11), and hence *no* solutions.

Now, given any set A, B, C, $0 < A < B < C$, we first find G, H, K by (9). Then, if possible, we choose any T satisfying (11). From (10) we see $D = F - KT \leqq C - KT$ so that $C - KT$ is an upper bound for D. So we choose D such that

$$0 \leqq D \leqq C-KT . \tag{12}$$

To obtain E and F, we note from (10)

$$E = D+HT, \quad F = D+KT$$

and from (5) and (9)

$$\frac{Y}{X} = 1 - \frac{HG}{HT} \quad \text{or} \quad X : Y = T : T-G \tag{13}$$

will be a solution in the Fehr sense.

Since D, $D + HT$, $D + KT$ are the number of eggs each boy, respectively, sells at Y cents apiece, we see at once that if D and T have a factor greater than 1 in common, say n*, with $n > A-D$, we will have a solution in the salesman sense with n the number of eggs in a "lot", the price per lot being nY. If $n \leqq A-D$, a solution

*Any integer n is a factor of 0.

in the Fehr sense would not admit interpretation in the salesman sense since the remainder of eggs sold by each boy at X cents apiece would yield at least one more "lot".

This interpretation appears forced in lieu of further mathematical changes in (3). For example, in the original problem of 10, 30, and 50 eggs, one solution is $D = 7$, $E = 28$, $F = 49$, $X = 21$, $Y = 1$, but if a "lot" is defined to be 7 eggs, we can say each boy sold 1, 4, and 7 "lots", respectively, at 7 cents a "lot". At best this may well be called a waste of words.

Fortunately, in such a solution further mathematical simplification of the expression (3) allows a "more primitive" solution whose interpretation is quite different from the original one. We consider (13) for two classes of values for T and G: (a) T prime to G; (b) T not prime to G.

(a) If $(T|G) = 1$, then the smallest permissible value of X is T. Consequently, n divides X and expression (3) can be reduced by dividing each of the three binomials by n as follows: divide n into the quantity factor of the first term and into the price factor X in the second term. This yields a "more primitive" solution and a new salesman solution.

(b) If $(T|G) = d > 1$, then the smallest permissible value of X is T/d. In this case n may be equal to or greater than X. Here we may get several "more primitive" solutions and salesman solutions. The following examples illustrate the various cases mentioned.

Example 1. $A = 10$, $B = 30$, $C = 50$.

Since $B-A = 20$, $C-A = 40$, and therefore $G = 20$, $H = 1$, $K = 2$, we can choose T, $20 < T \leqq 50/2$, and D, $0 \leqq D \leqq 50 - 2T$ as follows:

$T = 21,\ 0 \leqq D \leqq 8,$	9 solutions	
$T = 22,\ 0 \leqq D \leqq 6,$	7 "	
$T = 23,\ 0 \leqq D \leqq 4,$	5 "	25 solutions
$T = 24,\ 0 \leqq D \leqq 2,$	3 "	in Fehr sense
$T = 25,\ 0 \leqq D \leqq 0,$	1 solution	

For instance, when $T = 23$, let us take $D = 3$; then we have $E = 26$, $F = 49$, $X:Y = 23:3$, and (3) becomes

$$3(3) + 7(23) = 26(3) + 4(23) = 49(3) + 1(23) = 170.$$

When $T = 22$ and $D = 0$, we have $E = 22$, $F = 44$, $X:Y = 22:2 = 11:1$, and (3) becomes

$$0(2) + 10(22) = 22(2) + 8(22) = 44(2) + 6(22) = 220$$

or

$$0(1) + 10(11) = 22(1) + 8(11) = 44(1) + 6(11) = 110\ .$$

This solution a la Fehr corresponds to *two* solutions a la salesman:

$n = 11$, 1 cent for each lot of 11, 1 cent for each odd egg
$$0(1) + 10(1) = 2(1) + 8(1) = 4(1) + 6(1) = 10$$

$n = 22$, 2 cents for each lot of 22, 1 cent for each odd egg
$$0(2) + 10(1) = 1(2) + 8(1) = 2(2) + 6(1) = 10.$$

For $T = 21$ and $D = 7$, we have $E = 28$, $F = 49$, $X:Y = 21:1$ and (3) becomes

$$7(1) + 3(21) = 28(1) + 2(21) = 49(1) + 1(21) = 70.$$

The corresponding salesman solution, with $n = 7$, is

$$1(1) + 3(3) = 4(1) + 2(3) = 7(1) + 1(3) = 10.$$

There are 8 solutions in the salesman sense, corresponding to

T	21	21	22	22	23	24	24	25
D	0	7	0	0	0	0	0	0
n	21	7	11	22	23	24	12	25

Example 2. $A = 11$, $B = 30$, $C = 50$

Here we have $B\text{-}A = 19$, $C\text{-}A = 39$, $G = 1$, $H = 19$, and $K = 39$. However,

$$\frac{C}{K} = \frac{50}{39} < 2$$

and hence we can not have a T that satisfies (11). There are no solutions in the Fehr sense, and therefore none in the salesman sense.

Example 3. $A = 25$, $B = 51$, $C = 103$

Since $B\text{-}A = 26$, $C\text{-}A = 78$, and therefore $G = 26$, $H = 1$, $K = 3$, we can choose T, $26 < T < 103/3$, and D, $0 \leqq D \leqq 103 - 3T$ as follows:

$T = 27,\ 0 \leqq D \leqq 22,$	23 solutions		
$T = 28,\ 0 \leqq D \leqq 19,$	20 "		
$T = 29,\ 0 \leqq D \leqq 16,$	17 "	100 solutions in Fehr sense	
....	....		
$T = 34,\ 0 \leqq D \leqq 1\ ,$	2 "		

For instance, when $T = 30$, let us take $D = 12$, then we have $E = 42$, $F = 102$, $X:Y = 15:2$, and (3) becomes

$$12(2) + 13(15) = 42(2) + 9(15) = 102(2) + 1(15) = 219.$$

This Fehr solution does not lead to a salesman solution because T and D do not have a common factor n, $n > A-D = 13$.

For $T = 27$ and $D = 18$, we have $E = 45$, $F = 99$, $X:Y = 27:1$ and (3) becomes

$$18(1) + 7(27) = 45(1) + 6(27) = 99(1) + 4(27) = 207.$$

The corresponding salesman solution, with $n = 9$, is

$$2(1) + 7(3) = 5(1) + 6(3) = 11(1) + 4(3) = 23.$$

In closing we note that if all zero solutions are ruled out, the number of solutions is greatly reduced. Some extension of this work is now being considered and will be reported at a later date.

Lebanon Valley College

Milford, Texas
November 23, 1954

Dear Doctor James:

Sound mathematics at any level is an intellectual occupation. At any required level no worthwhile article for that reader can be so simple that it can be assimilated at one reading. Pencil and paper judiciously used are necessary adjuncts to the visual work of reading. If a man is not willing to do that, he is, as far as mathematics is concerned, certainly not an amateur, not even a dilettante, but merely a dabbler. The charm of mathematics is the gradually increasing comprehension that comes with intellectual effort and that gives one the satisfaction of accomplishment.

At this point may I digress and bring up a subject in which I feel perhaps too deeply concerned. I refer to the fifth paragraph of your published letter to Dr. Dimmick. My contention is this. When the "Queen of the Sciences" first appeared it was widely read because mathematics at the lower levels was taught by well trained men who could and did inspire their students with their grasp of the subject and with their friendly and encouraging interest in their progress. This situation has changed.

Today the requirements for teachers of mathematics at the pre-college level are determined by educators interested not in the subject but in methodology and in novelties. Merely investigate the intellectual level at which mathematics is taught, at least here in the southwest. Ad after ad appears in the Texas and Oklahoma papers, requesting teachers of high school mathematics and physics.

In general they read "Wanted, man for football (or basketball) coach. Must teach mathematics (and/or physics)". I even saw one that required, in addition, the driving of a school bus. To get a realistic approach to the situation, I personally called on a superintendent in a small town in reply to his ad for a teacher of H. S. Mathematics. He took one look at me and in essence he said: "Of course the man we want should know a little about math but what I really need is a basketball coach who will give me a winning team." I sat in an algebra class at another 'High School' in a small town. One problem the instructor was 'explaining' resolved itself into the equation $x^2 = x$. He divided through by x, came up with answer $x = 1$, and that was that. To such men Euclid's Axioms are still self evident truths. Students from such high schools must be admitted to the state colleges, so we have articles in mathematical magazines on such subjects as: *Do We Really Teach College Algebra? Teacher Education in Algebra, The Poorly Prepared College Student,* etc. General Electric advertise "2 + 2 = 4 is no longer enough". And still I don't believe that our gifted mathematicians realize the mathematical immaturity of the great majority of our citizens.

It is only through the united educational efforts of the mathematicians that this situation can ever be remedied. Articles wont accomplish it. It will grow steadily worse, and hence in this $E = mc^2$ age of ours, the 5th paragraph of your letter should convey a dire warning, and should awaken mathematicians to the danger of this situation, not for mathematicians alone but for the future of this country.

Your magazine certainly can do wonderful pioneer work, not only in interesting young men of intelligence in mathematics, but also in combating this increasing tendency in our elementary schools to teach mathematics at an ever decreasing level.

Sincerely yours,

(Signed) L. E. Diamond